Comment on "Simple Measure for Complexity"

James P. Crutchfield, David P. Feldman,* and Cosma Rohilla Shalizi†
Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, NM 87501
Electronic address: {chaos,dpf,shalizi}@santafe.edu
(February 20, 2000)

We critique the measure of complexity introduced by Shiner, Davison, and Landsberg in Ref. [1]. In particular, we point out that it is over-universal, in the sense that it has the same dependence on disorder for structurally distinct systems. We then give counterexamples to the claim that complexity is synonymous with being out of equilibrium: equilibrium systems can be structurally complex and nonequilibrium systems can be structurally simple. We also correct a misinterpretation of a result given by two of the present authors in Ref. [2].

Santa Fe Institute Working Paper 99-06-040

In Ref. [1], Shiner, Davison, and Landsberg introduce a two-parameter family $\Gamma_{\alpha\beta}$ of complexity measures:

$$\Gamma_{\alpha\beta} \equiv \Delta^{\alpha}(1-\Delta)^{\beta} , \quad (1)$$

where

$$\Delta \equiv \frac{S}{S_{\max}} . \quad (2)$$

The quantity Δ is called the "disorder", S is the Boltzmann-Gibbs-Shannon entropy of the system, and $S_{\max}$ its maximum possible entropy—taken to be equal to the equilibrium thermodynamic entropy. For $\alpha, \beta > 0$, $\Gamma_{\alpha\beta}$ satisfies the widely accepted "one-hump" criterion for statistical complexity measures—the requirement that any such measure be small for both highly ordered and highly disordered systems [3–7]. The approach to complexity measures taken by Shiner, Davison, and Landsberg [1] is similar to that of Lòpez-Ruiz, Mancini, and Calbet [8]. In both Refs. [1] and [8] the authors obtain a measure of complexity satisfying the one-hump criterion by multiplying a measure of "order" by a measure of "disorder".

We welcome this addition to the literature on complexity measures and are pleased to see a variety of complexity measures compared and examined critically. However, there are several aspects of Ref. [1] upon which we would like to comment.

First, despite satisfying the one-hump criterion, it is not clear that $\Gamma_{\alpha\beta}$ is a measure of *complexity*. $\Gamma_{\alpha\beta}$ is a quadratic function of a measure of distance from thermodynamic equilibrium, as the authors note on p. 1461. This has three consequences:

1. As pointed out in Ref. [9], this type of complexity measure is over-universal in the sense that it has the same dependence on disorder for structurally distinct systems. Eq. (1) makes it clear that, despite the claims of Shiner et al. to the contrary, all systems with the same disorder Δ have the same $\Gamma_{\alpha\beta}$.

2. Since $S_{\max}$ is taken to be the equilibrium entropy of the system, $\Gamma_{\alpha\beta}$ vanishes for *all* equilibrium systems: "'Complexity' vanishes ... if the system is at equilibrium" [1, p. 1461]. Due to this $\Gamma_{\alpha\beta}$ does not distinguish between two-dimensional Ising systems at low temperature, high temperature, or the critical temperature. All of these systems are at equilibrium and hence have vanishing $\Gamma_{\alpha\beta}$. However, they display strikingly different *degrees* of structure and organization. Nor does $\Gamma_{\alpha\beta}$ distinguish between the many different *kinds* of organization observed in equilibrium [10]—between, say, ideal gases, the long-range ferromagnetic order of low-temperature Ising systems, the orientational and spatial order of the many different liquid crystal phases [11], and the intricate structures formed by amphiphilic systems [12]. All of these systems are in equilibrium, but they (presumably) have very different complexities.

3. We have just seen that equilibrium should not be taken to indicate an absence of complexity. Conversely, not all systems out of equilibrium are complex. For example, consider a paramagnet, a collection of two-state spins that are not coupled. If this system is pumped so that it's out of equilibrium, a larger percentage of the spins will be in their higher energy states. Nevertheless, there is still no spatial structure or ordering in the system; the spins are still completely uncorrelated. However, the complexity measure of Shiner et al. will be nonzero for this very simple system. While $\Gamma_{\alpha\beta}$ vanishes for systems at "maximal distance from equilibrium" Ref. [1, p. 1461], all other systems displaced from equilibrium have non-vanishing com-

*Permanent address: College of the Atlantic, 105 Eden Street, Bar Harbor, Maine 04609.
†Permanent address: Physics Department, University of Wisconsin, Madison, WI 53706.

plexity by virtue of the $1 - \Delta$ term in Eq. (1). It does not seem reasonable to us to require that *any* system partially out of equilibrium have positive complexity.

In summary, then, we argue that whether or not a system is in equilibrium in and of itself says little about the system's structure, pattern, organization, or symmetries. Equilibrium systems can be complex, nonequilibrium systems can be simple, and vice versa. Since $\Gamma_{\alpha\beta}$ is defined in terms of a "distance from equilibrium" $1 - \Delta$, we feel that it cannot capture structural complexity.

Second, we are confused by Ref. [1]'s calculation of Γ_{11} for equilibrium Ising systems on p. 1462. If the system is at equilibrium, then the disequilibrium term $1-\Delta$ should vanish, leading to a vanishing Γ_{11}. Perhaps the authors are using a uniform distribution rather than the thermodynamic equilibrium distribution in their calculation of $S_{\max}$.

Third, Ref. [1] appears to have misinterpreted our earlier work on the statistical complexity of one-dimensional spin systems [2,13]. On p. 1462, Ref. [1] identifies the statistical complexity C_μ [5,14] with zero-coupling ($J = 0$) disorder Δ. At a minimum, this interpretation is not consistent dimensionally, since C_μ has the units of entropy (bits), while Δ is a dimensionless ratio. More crucially, however, Ref. [1] conflates the *definition* of C_μ, which does not make C_μ a function solely of the system's entropy, with *a particular equation* for C_μ (Eq. (8) of Ref. [2]) correct within a strictly delimited range of validity [2,13]. Further, Ref. [1] draws an inaccurate conclusion based on that equation. For nearest-neighbor Ising systems Refs. [2] and [13] show that $C_\mu = H(1)$, the entropy of spin blocks of length one. Contrary to the statement in Ref. [1], $H(1)$ is not the same as the entropy of noninteracting spins—i.e., of paramagnetic spin systems, those with $J = 0$.

Finally, Ref. [1] states that thermodynamic depth [15] belongs to the family of complexity measures that are single-humped functions of disorder. However, two of us recently pointed out that thermodynamic depth is an increasing function of disorder [16].

In summary, we have argued here and elsewhere [13,14] that a useful role for statistical complexity measures is to capture the structures—patterns, organization, regularities, symmetries—intrinsic to a process. Ref. [9] emphasizes that defining such measures solely in terms of the one-hump criterion—say, by multiplying "disorder" by "one minus disorder"—is insufficient to this task. Introducing an arbitrary parameterization of this product—e.g. via α and β in Eq. (1)—does not help the situation. A statistical complexity measure that is a function only of disorder is not adequate to measure structural complexity, since it is unable to distinguish between structurally distinct configurations with the same disorder.

This work was supported at the Santa Fe Institute under the Computation, Dynamics and Inference Program via ONR grant N00014-95-1-0975 and by Sandia National Laboratory. We thank an anonymous referee for several helpful comments.

---

[1] J. S. Shiner, M. Davison, and P. T. Landsberg. Simple measure for complexity. *Phys. Rev. E*, 59:1459–1464, 1999.

[2] J. P. Crutchfield and D. P. Feldman. Statistical complexity of simple one-dimensional spin systems. *Phys. Rev. E*, 55(2):1239R–1243R, 1997.

[3] P. Grassberger. Toward a quantitative theory of self-generated complexity. *Intl. J. Theo. Phys.*, 25(9):907–938, 1986.

[4] K. Lindgren and M. G. Norhdal. Complexity measures and cellular automata. *Complex Systems*, 2(4):409–440, 1988.

[5] J. P. Crutchfield and K. Young. Inferring statistical complexity. *Phys. Rev. Lett.*, 63:105–108, 1989.

[6] M. Gell-Mann and S. Lloyd. Information measures, effective complexity, and total information. *Complexity*, 2(1):44–52, 1996.

[7] R. Badii and A. Politi. *Complexity: Hierarchical structures and scaling in physics.* Cambridge University Press, Cambridge, 1997.

[8] R. Lopez-Ruiz, H. L. Mancini, and X. Calbet. A statistical measure of complexity. *Phys. Lett. A*, 209:321–326, 1995.

[9] D. P. Feldman and J. P. Crutchfield. Measures of statistical complexity: Why? *Phys. Lett. A*, 238:244–252, 1998.

[10] P. M. Chaikin and T. C. Lubensky. *Principles of Condensed Matter Physics.* Cambridge University Press, Cambridge, 1995.

[11] P. J. Collings and M. Hird. *Introduction to Liquid Crystals: Chemistry and Physics.* Taylor and Francis, London, 1997.

[12] G. Gompper and M. Schick. *Self-assembling Amphiphilic Systems*, volume 16 of *Phase transitions and critical phenomena.* Academic Press, San Diego, 1994.

[13] D. P. Feldman and J. P. Crutchfield. Discovering noncritical organization: Statistical mechanical, information theoretic, and computational views of patterns in simple one-dimensional spin systems. *J. Stat. Phys.*, 1998. submitted; Santa Fe Institute Working Paper 98-04-026; http://www.santafe.edu/projects/CompMech/papers/DNCO.html.

[14] J. P. Crutchfield. The calculi of emergence: Computation, dynamics, and induction. *Physica D*, 75:11–54, 1994.

[15] S. Lloyd and H. Pagels. Complexity as thermodynamic depth. *Ann. Phys.*, 188:186–213, 1988.

[16] J. P. Crutchfield and C. R. Shalizi. Thermodynamic depth of causal states: Objective complexity via minimal representations. *Phys. Rev. E*, 59(1):275–283, 1998.