Introduction to Chaos and Dynamical Systems

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Overview and Outline

1. Quick Introduction
2. Chaos I: Introduction to Chaos
3. Chaos II: Bifurcations and Universality
4. Chaos III: Strange Attractors
5. Conclusion

- Main Goal: Give a solid, not too technical introduction to the key phenomena and insights from the study of chaos and dynamical systems.
- Focus on relations of chaos and dynamical systems to complex systems more generally.
- Please ask questions during (and after) my lecture.
- These notes are based on a week-long series of lectures I have given at SFI's Complex Systems Summer School since 2004.

Complex Systems?

Melanie/Chris have given a few ways to think about complex systems:

1. Ingredients: Dynamics, Information, Computation, Evolution and Learning, ...
2. Methods and Models: Statistical Physics, Agent-based models, Networks, Chaos and dynamics, ...
3. Phenomena: Immune system, ecosystems, economies, auction markets, evolutionary systems, the brain, natural computation, ...
4. Theoretical: General principles?
5. A particular interdisciplinary mix, style, and point of view.

However one thinks of complex systems, Chaos and Dynamical Systems play a role.

Chaos: The *Longue Durée*

Thoughts on how to think about chaos:

"We take the emergence of 'chaos' as a science of nonlinear phenomena... as a vast process of sociodisciplinary convergence and conceptual reconfiguration.... In order to come up with an exhaustive historical analysis of these origins [of "nonlinear science"] one needs to be able to deal at once with domains as varied as fluid mechanics, parts of engineering, and population dynamics."

Aubin and Dahan-Dalmedico, *Historia Mathematica* 29 (2002), 1-67. doi:10.1006/hmat.2002.2351

They refer to chaos as having an "**ample and bushy genealogy**."

Chaos is not a sudden revolution.

Complexity: The *Longue Durée*?

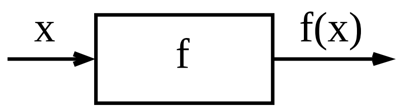
- I believe much the same can be said about Complex Systems.
- There are many different streams of thought that flow together to form the study of Complex Systems: Chaos/Dynamical Systems, Genetic Algorithms/A-life, Economics, and so on.
- The confluence of these streams is not a unitary discipline or a coherent theory, but a “sociodisciplinary convergence and conceptual reconfiguration.”
- Complex Systems has a tangled genealogy. But one of the deepest roots is the study of chaos and dynamical systems.

Chaos I: An Overview of Dynamical Systems and Chaos

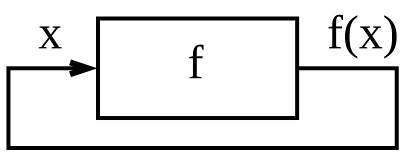
- A **Dynamical System** is any system that changes over time according to some rule:
  - A differential equation
  - A system of differential equations
  - Iterated functions
  - Cellular automata
- The goal of this brief introduction is to define a handful of terms, define chaos and sensitive dependence on initial conditions, and briefly discuss some of its implications.
- I will focus on iterated functions.
- Let's start with an example.

Example: Iterating the squaring rule, $f(x) = x^2$

- Consider the function $f(x) = x^2$. What happens if we start with a number and repeatedly apply this function to it?
- E.g., $3^2 = 9$, $9^2 = 81$, $81^2 = 6561$, etc.
- The iteration process can also be written $x_{n+1} = x_n^2$.
- In this is example, the initial value 3 is the **seed**, often denoted x_0.
- The sequence $3, 9, 81, 6561, \cdots$ is the **orbit** or the **itinerary** of 3.
- Picture the function as a "box" that takes x as an input and outputs $f(x)$:



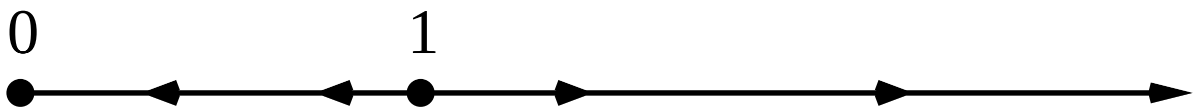
- Iterating the function is then achieved by feeding the output back to the function, making a feedback loop:



The squaring rule, continued

In dynamics, we are usually interested in the long-term behavior of the orbit, not in the particulars of the orbit.

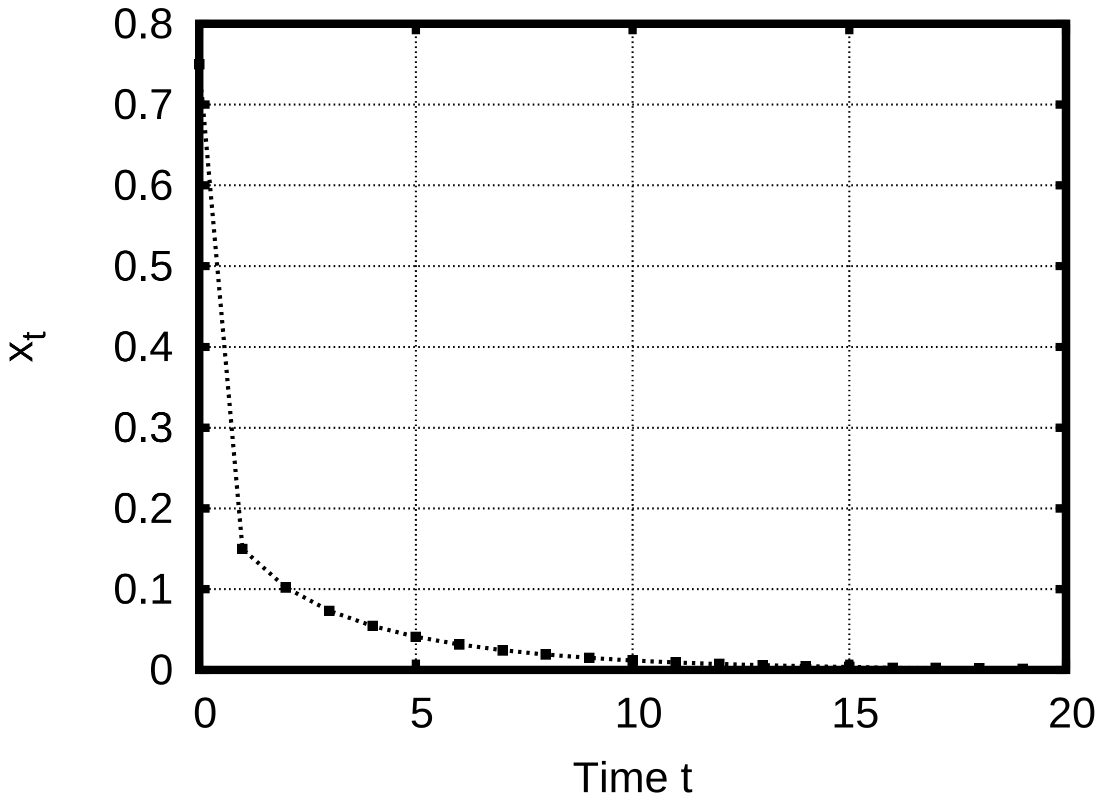
- The seed 3 tends toward infinity—it gets bigger and bigger.
- Any $x_0 > 1$ will tend toward infinity.
- If $x_0 = 1$ or $x_0 = 0$, then the point never changes. These are fixed points.
- If $0 \leq x_0 < 1$, then x_0 approaches 0.
- We can summarize this with the following diagram:



- 0 and 1 are both **fixed points**
- 0 is a **stable** or **attracting** fixed point
- 1 is an **unstable** or **repelling** fixed point

Logistic Equation

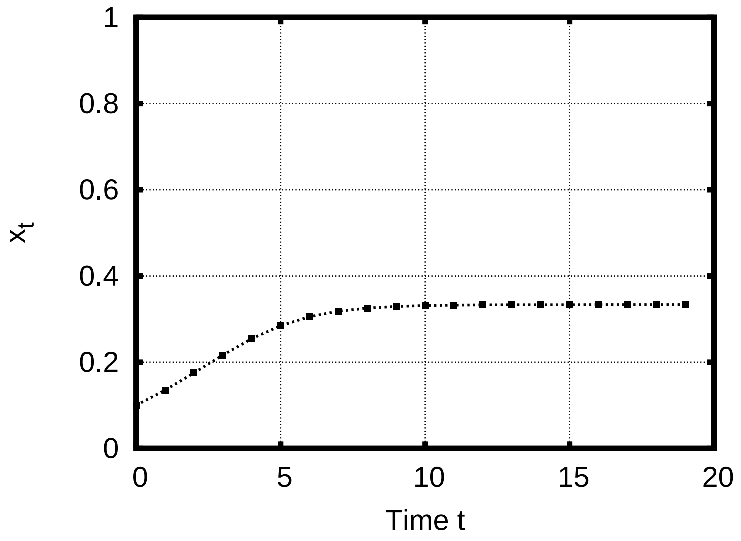
- Logistic equation: $f(x) = rx(1-x)$.
- A simple model of resource-limited population growth.
- The population x is expressed as a fraction of the carrying capacity. $0 \leq x \leq 1$.
- r is a parameter—the growth rate—that we will vary.
- Let's first see what happens if $r = 0.8$.

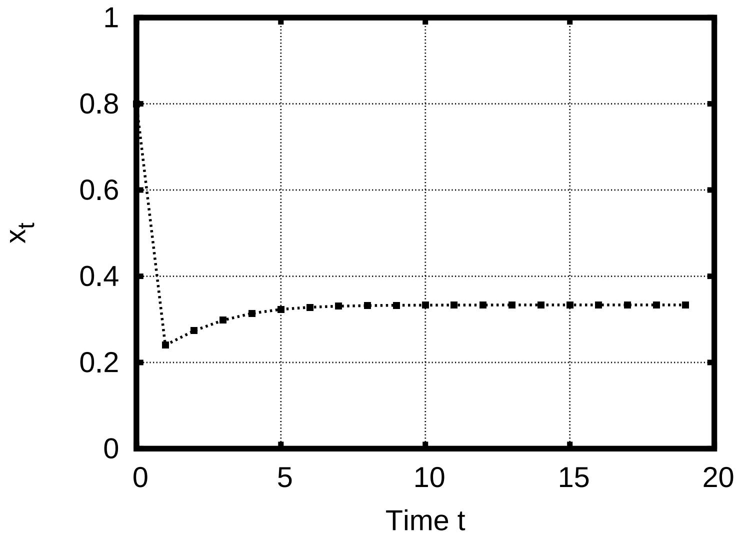


- You can make your own plots at `http://hornacek.coa.edu/dave/Chaos/`.
- 0 is an attracting fixed point.

Logistic Equation, $r = 1.5$

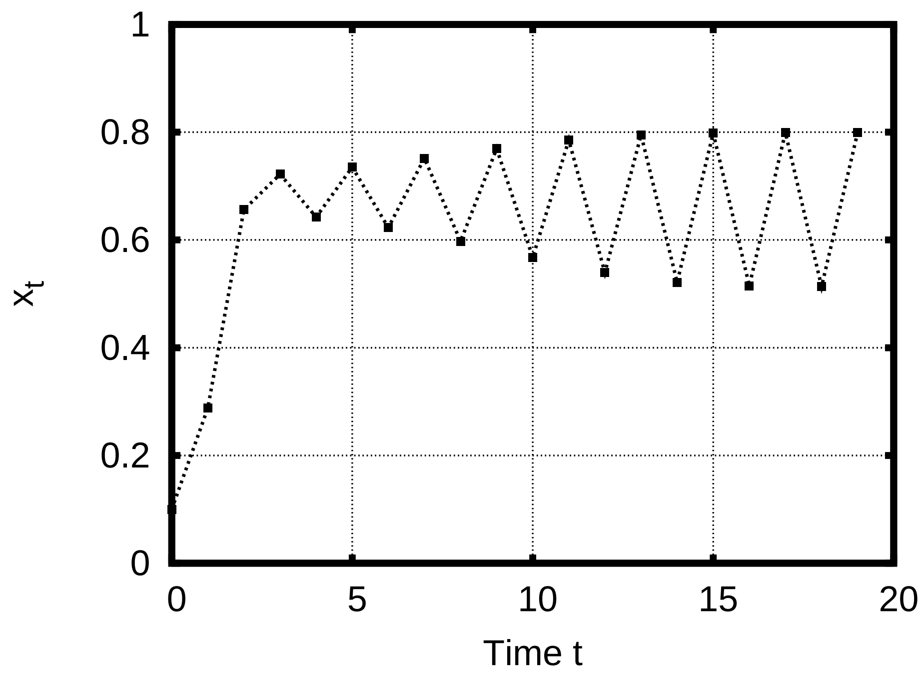
- Logistic equation, $r = 1.5$.





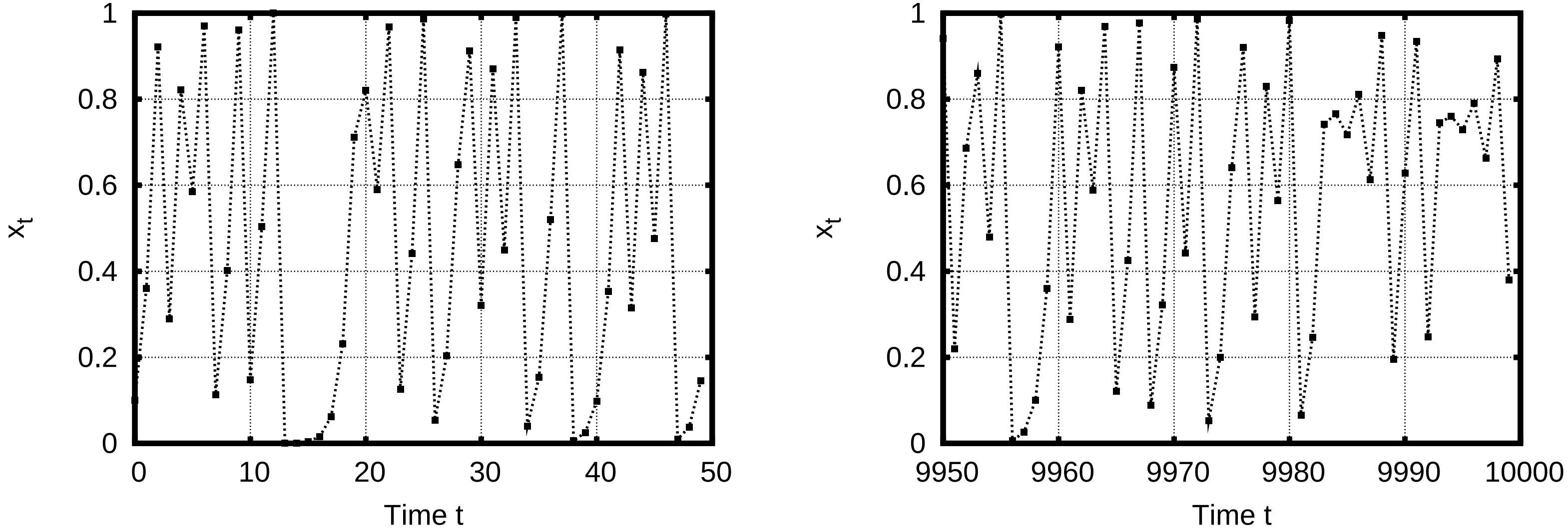
- All initial conditions are pulled toward 0.33.
- 0.33 is an attracting fixed point.

Logistic Equation, $r = 3.2$



- Logistic equation, $r = 3.2$.
- Initial conditions are pulled toward a **cycle** of period 2.
- The orbit oscillates between 0.513045 and 0.799455.
- This cycle is an attractor. Many different initial conditions get pulled to it.

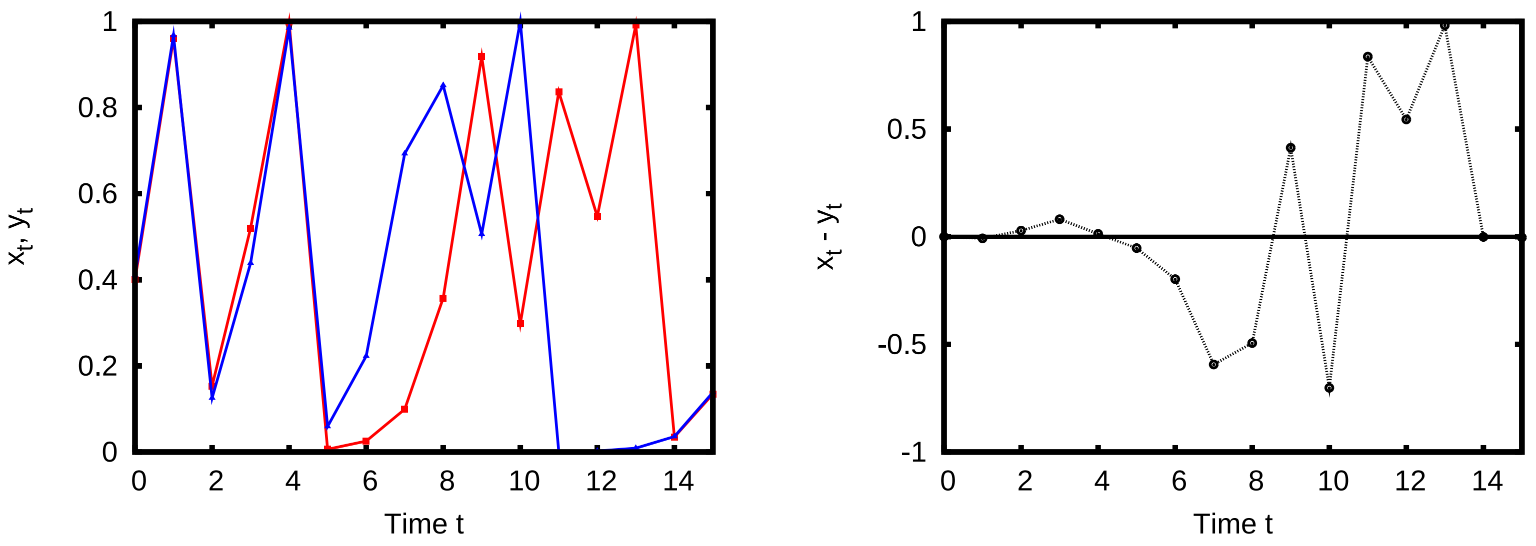
Logistic Equation, $r = 4.0$



- Logistic equation, $r = 4.0$.
- What's going on here?!
- The orbit is not periodic. In fact, it never repeats.
- This is a rigorous result; it doesn't rely on computers.
- What happens if we try different initial conditions?

Different Initial conditions: $r = 4.0$

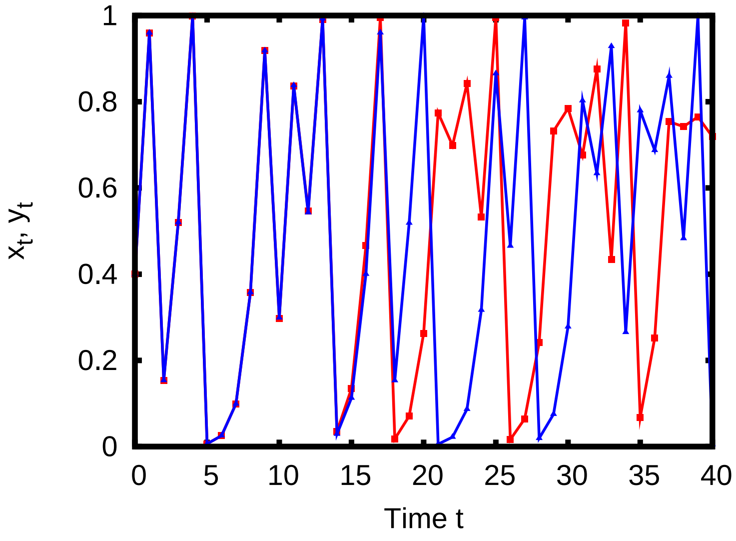
- Two slightly different initial conditions, $x_0 = 0.4$ and $x_0 = 0.41$.

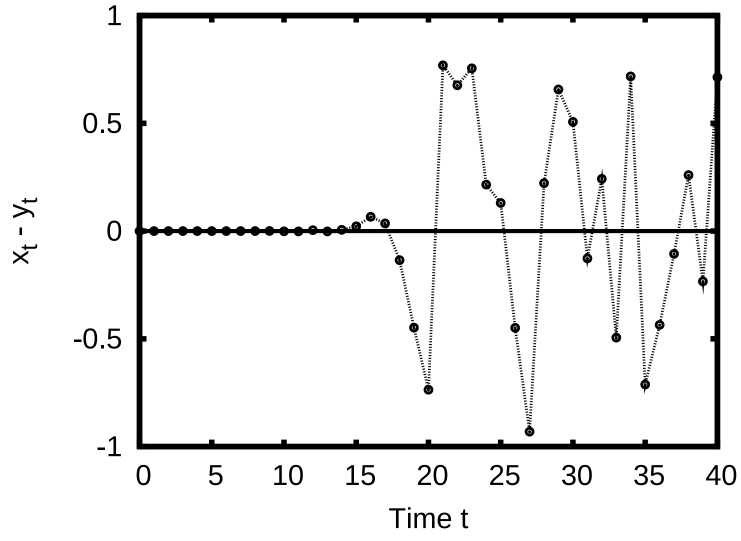


- The right graph plots the difference between the two orbits on the left
- Note that the difference between the two orbits grows.
- Can think of the blue as the true values, and the red as the predicted values.
- The plot on the right can be viewed as prediction error over time.
- How can we improve our predictions?

Sensitive Dependence on Initial Conditions

- Two different initial conditions, $x_0 = 0.4$ and $x_0 = 0.4000001$.

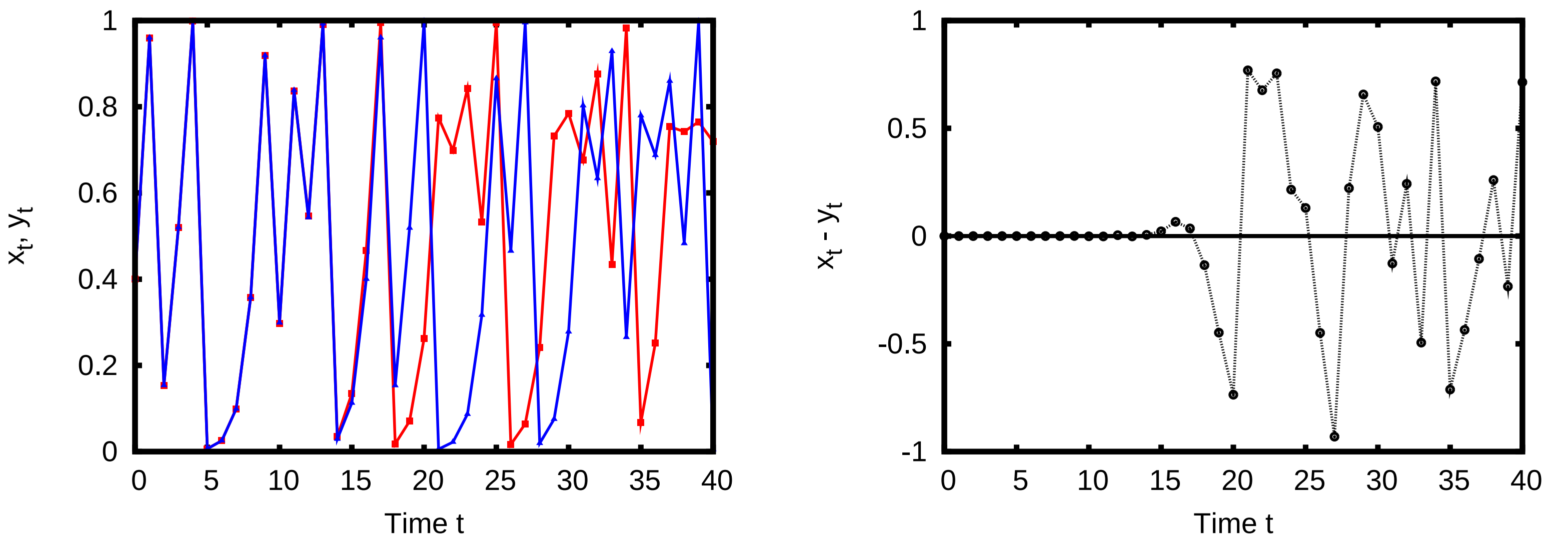




- The two initial conditions differ by one part in one million
- The orbits differ significantly after around 20 iterations, whereas before they differed after around 4 iterations.
- Increasing the accuracy of the initial condition by a factor of 10^5 allow us to predict the outcome 5 times further.
- Bummer.

Sensitive Dependence on Initial Conditions

- Two different initial conditions, $x_0 = 0.4$ and $x_0 = 0.4000001$.



- Thus, for all practical purposes, this system is unpredictable, even though it is deterministic.
- This phenomena is known as **Sensitive Dependence on Initial Conditions**, or, more colloquially, **The Butterfly Effect**.
- Arbitrarily small differences in initial conditions grow to become arbitrarily large.

Definition of Sensitive Dependence on Initial Conditions

- A dynamical system has sensitive dependence on initial conditions (SDIC) if arbitrarily small differences in initial conditions eventually lead to arbitrarily large differences in the orbits.

More formally

- Let X be a metric space, and let f be a function that maps X to itself: $f : X \mapsto X$.
- The function f has SDIC if there exists a $\delta > 0$ such that $\forall x_1 \in X$ and $\forall \epsilon > 0$, there is an $x_2 \in X$ and a natural number $n \in N$ such that $d[x_1, x_2] < \epsilon$ and $d[f^n(x_1), f^n(x_2)] > \delta$.
- In other words, two initial conditions that start ϵ apart will, after n iterations, be separated by a distance δ.

Definition of Chaos

There is not a 100% standard definition of chaos. But here is one of the most commonly used ones:

An iterated function is **chaotic** if:

1. The function is **deterministic**.
2. The system's orbits are **bounded**.
3. The system's orbits are **aperiodic**; i.e., they never repeat.
4. The system has **sensitive dependence on initial conditions**.

Other properties of a chaotic dynamical system ($f : X \mapsto X$) that are sometimes taken as defining features:

1. **Dense periodic points:** The periodic points of f are dense in X.
2. **Topological transitivity:** For all open sets $U, V \in X$, there exists an $x \in U$ such that, for some $n < \infty$, $f_n(x) \in V$. I.e., in any set there exists a point that will get arbitrarily close to any other set of points.

Chaos and Dynamical Systems: Selected References

There are many excellent references and textbooks on dynamical systems. Some of my favorites:

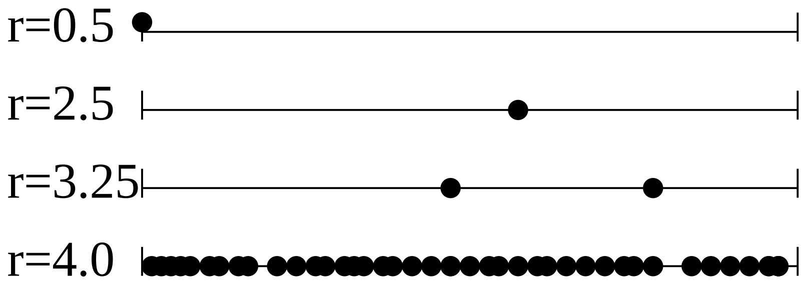
- Peitgen, et al. *Chaos and Fractals: New Frontiers of Science*. Springer-Verlag. 1992. *Huge (almost 1000 pages), and very clear. Excellent balance of rigor and intuition.*
- Gleick, *Chaos: Making a New Science*. Penguin Books. 1988. *Popular science book. Very good. Extremely well written and entertaining.*
- Steward, *Does God Play Dice?* (second edition.) Blackwell. 2002. *Another excellent book for a popular audience.*
- Strogatz. *Nonlinear Dynamics and Chaos*. Perseus Books Group. 2001.
- Smith. *Chaos: A Very Short Introduction*. Oxford. 2007.
- Feldman. *Chaos and Fractals: An Elementary Introduction*. Oxford. 2012.

Introduction to Chaos Part II: Bifurcations and Universality

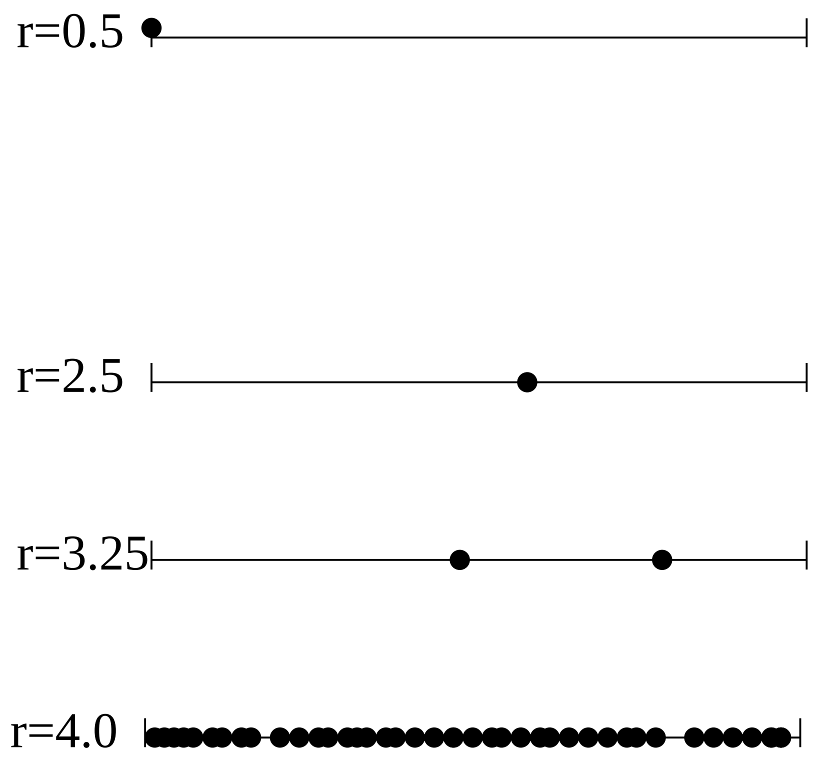
We have seen several possible long-term behaviors for the logistic equation:

1. $r = 0.5$: attracting fixed point at 0.
2. $r = 2.5$: attracting fixed point at 0.6.
3. $r = 3.25$: attracting cycle of period 2.
4. $r = 4.0$: chaos.

Graphically, we can illustrate this as follows:

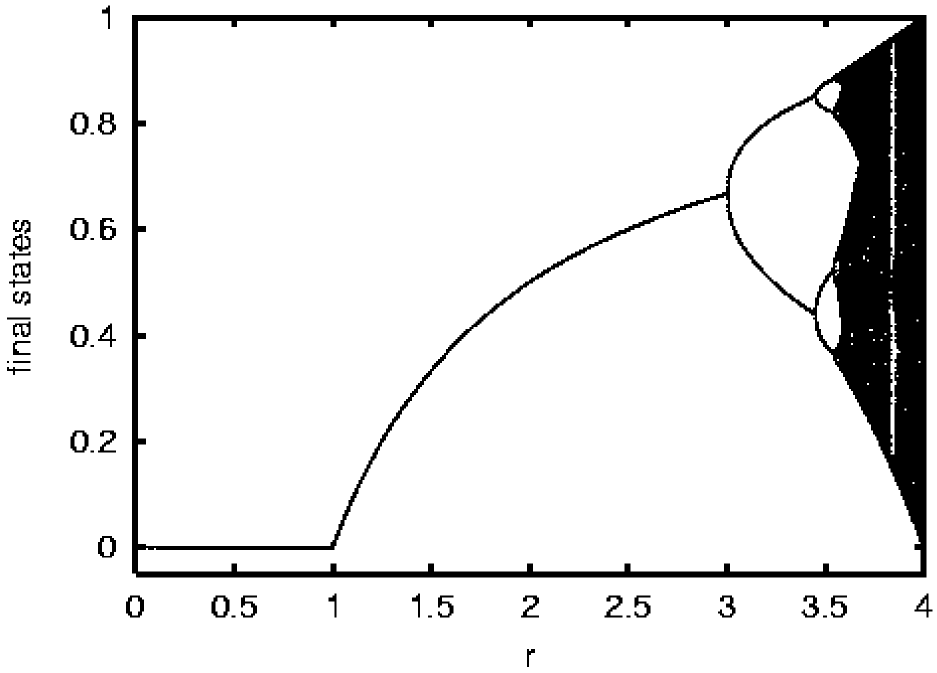


- I.e., for each r, iterate and plot the final x values as dots on the number line.
- What else can the logistic equation do??



- Do this for more and more r values and "glue" the lines together.
- Turn sideways and ...

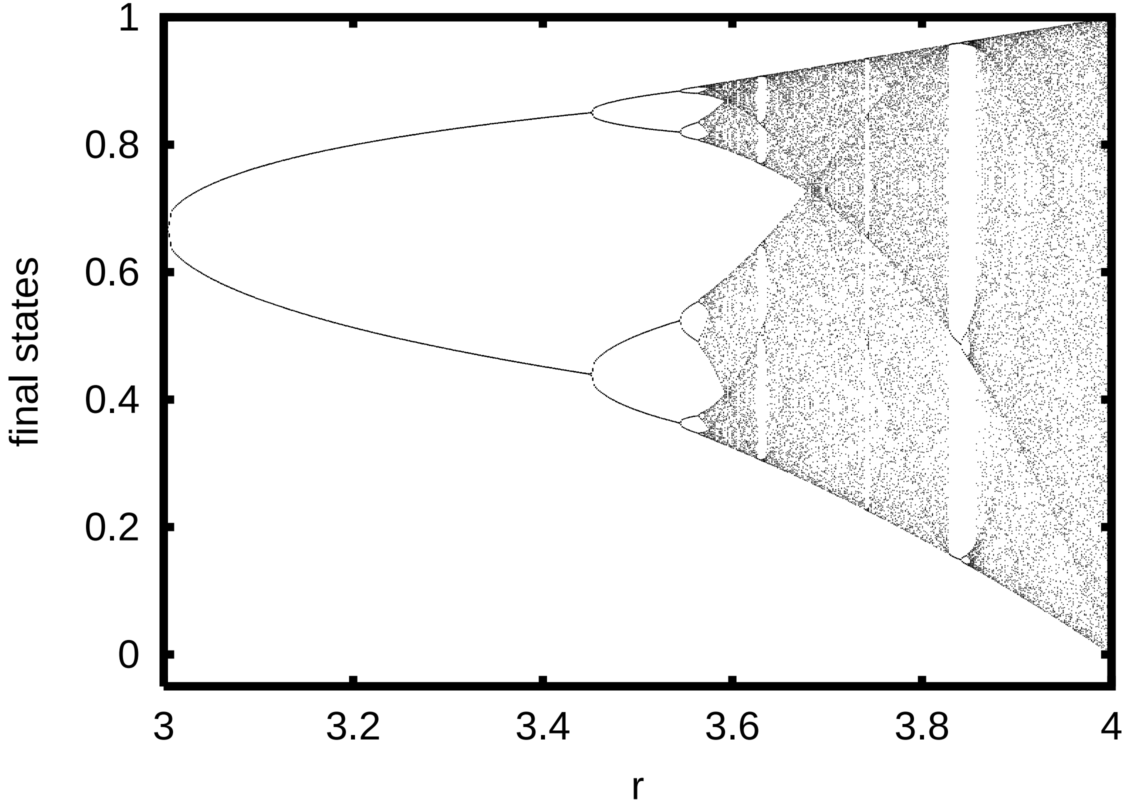
Bifurcation Diagram



- The bifurcation diagram shows the all the the possible long-term behaviors for the logistic map.
- $0 < r < 1$, the orbits are attracted to zero.
- $1 < r < 3$, the orbits are attracted to a non-zero fixed point.
- $3 < r < 3.45$, orbits are attracted to a cycle of period 2.
- Chaotic regions appear as dark vertical lines.

Bifurcation diagram, continued

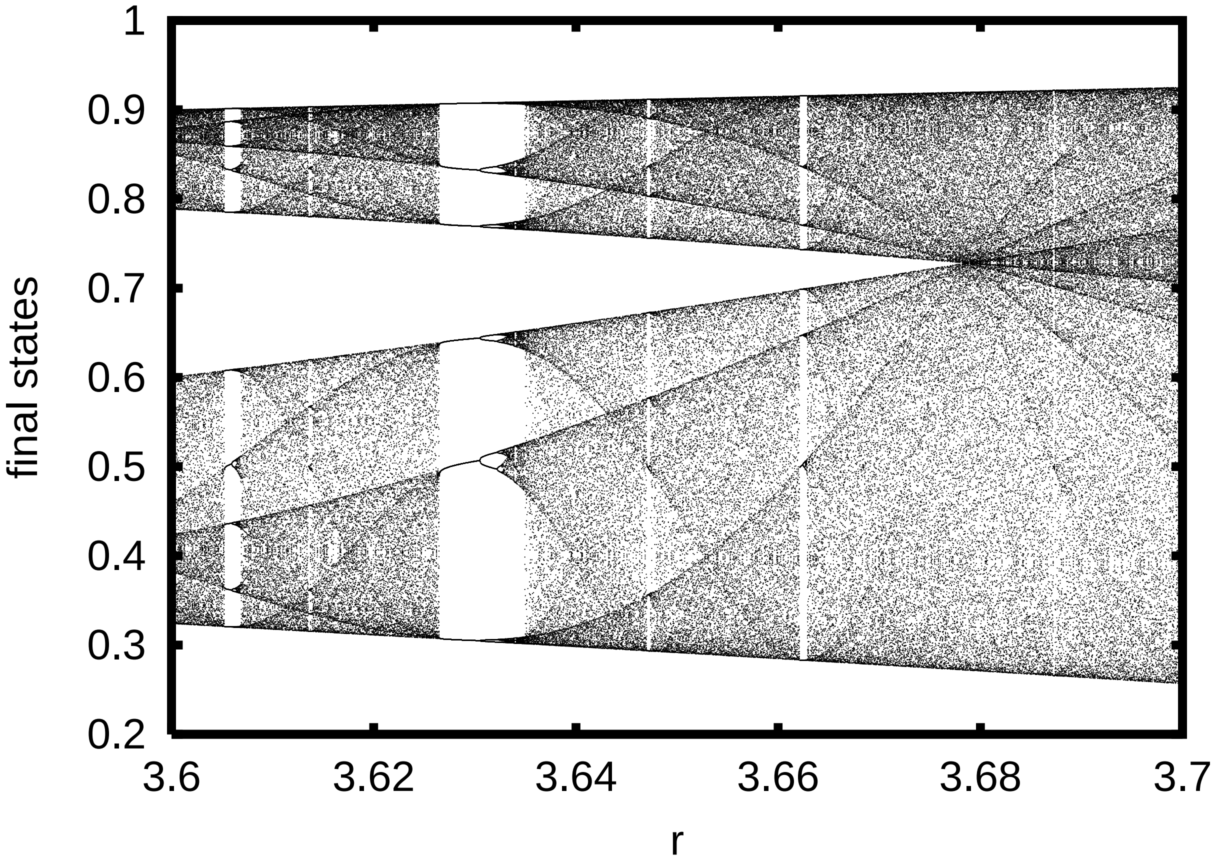
Let's zoom in on a region of the bifurcation diagram:



- The sudden qualitative changes are known as **bifurcations**.
- There are **period-doubling bifurcations** at $r \approx 3.45$, $r \approx 3.544$, etc.
- Note the window of period 3 near $r = 3.83$.

Bifurcation diagram, continued

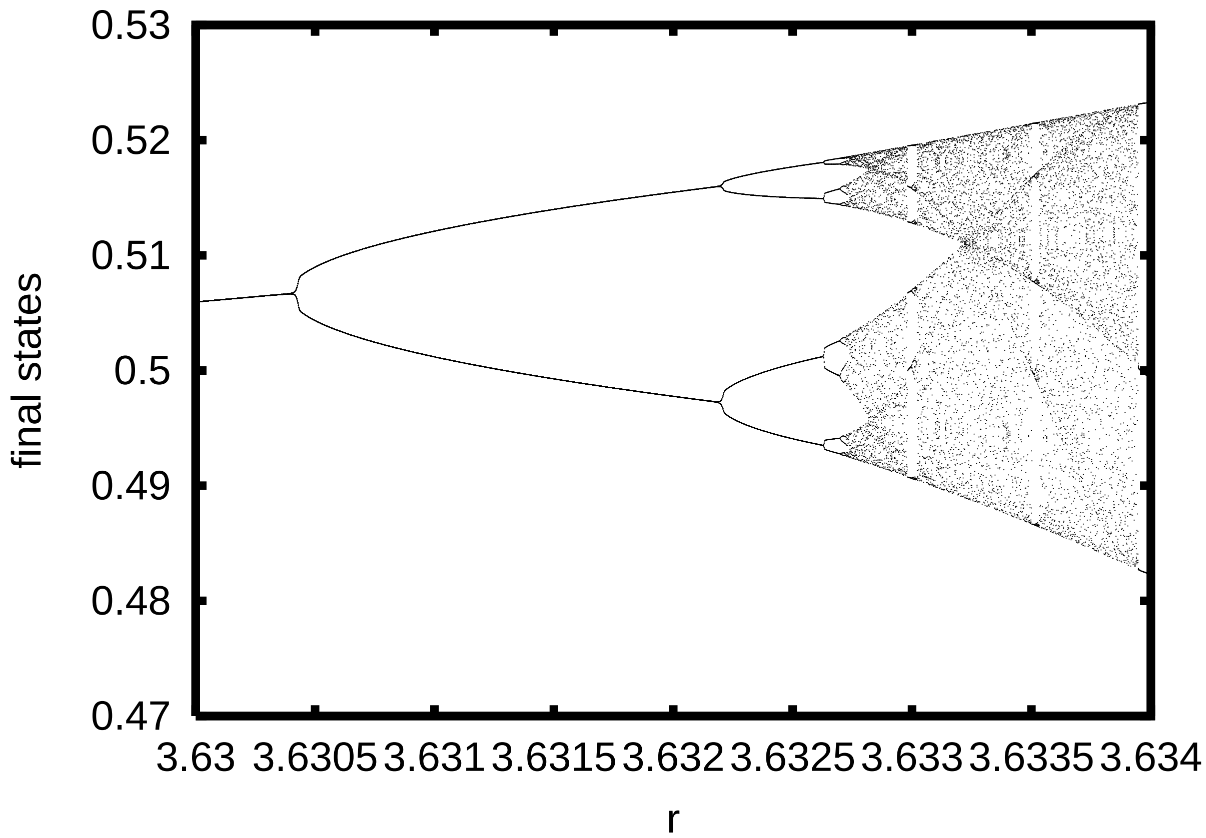
Let's zoom in again:



- Note the sudden changes from chaotic to periodic behavior.

Bifurcation diagram, continued

Let's zoom in once more:



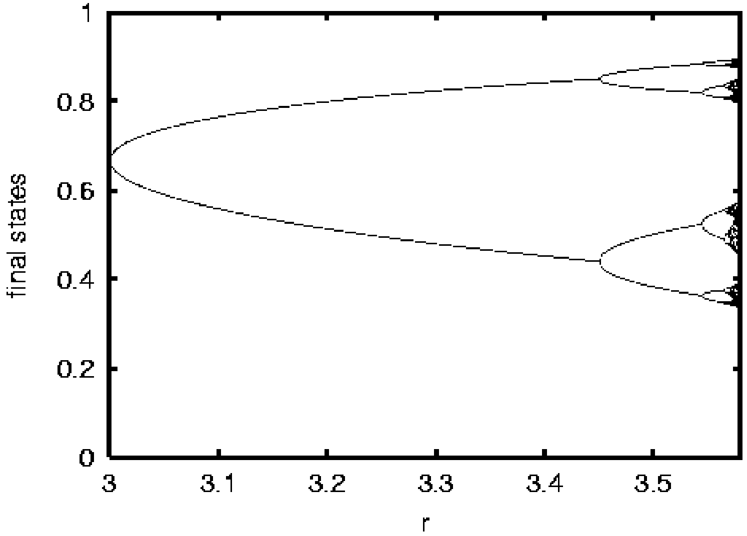
- Note the small scales on the vertical axis, and the tiny scale on the y axis.
- Note the self-similar structure. As we zoom in we keep seeing pitchforks.

Bifurcation Diagram Summary

- As we vary r, the logistic equation shuffles suddenly between chaotic and periodic behaviors, but the bifurcation diagram reveals that these transitions appear in a structured, or regular, way.
- This is an example of a sort of “order within chaos.”
- Bifurcations—a sudden, qualitative change in behavior as a parameter is continuously varied—is a generic feature of non-linear systems.
- In the next few slides we’ll examine one of the regularities in the bifurcation diagram: The **period-doubling route to chaos**.

Period-Doubling Route to Chaos

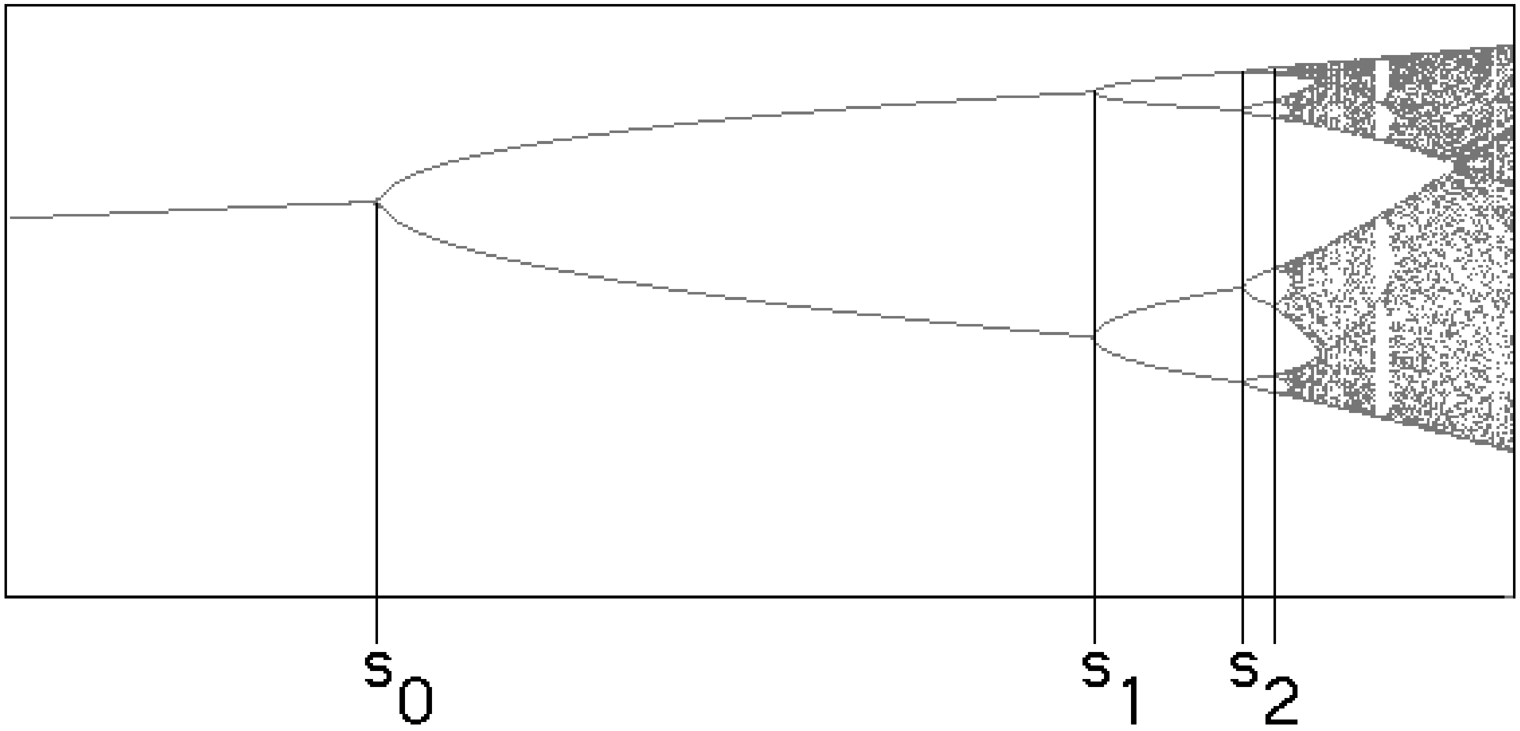
- As r is increased from 3, a sequence of period doubling bifurcations occur.



- At $r = r_\infty \approx 3.569945672$ the periods "accumulate" and the map becomes chaotic.
- For $r > r_\infty$ it has SDIC. For $r < r_\infty$ it does not.
- This is a type of **phase transition**: a sudden qualitative change in a system's behavior as a parameter is varied continuously.

Period-Doubling Route to Chaos: Geometric Scaling

- Let's examine the ratio of the lengths of the pitchfork tines in the bifurcation diagram.

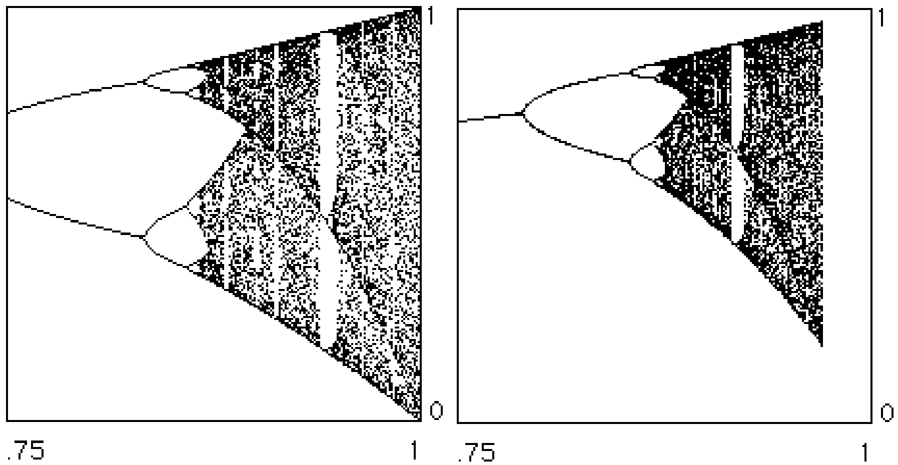


- The first ratio is: $\delta_1 = \frac{s_1 - s_0}{s_2 - s_1}$.
- The n$^{\mathrm{th}}$ ratio is: $\delta_n = \frac{s_n - s_{n-1}}{s_{n+1} - s_n}$.

Feigenbaum's Constant

- This ratio approaches a limit: $\lim_{n\to\infty} \delta_n = 4.669201609\ldots$. This is known as **Feigenbaum's constant** δ.
- This means that the bifurcations occur in a regular way.
- Amazingly, the value of δ is **universal**: it is the same for any period-doubling route to chaos!
- Figure Source: `http://classes.yale.edu/fractals/Chaos/Feigenbaum/Feigenbaum.html`

Universality



- The figure on the left is the bifurcation diagram for $f(x) = r\sin(\pi x)$.
- The figure on the right is the bifurcation diagram for $f(x) = \frac{27}{4} r x^2 (1-x)$.
- The bifurcation diagrams are very similar: **both have** $\delta \approx 4.6692$.
- Mathematically, things are constrained so that there is, in some sense, only one possible way for a system to undergo a period-doubling to chaos.
- Figure Source:

  `http://classes.yale.edu/fractals/Chaos/LogUniv/LogUniv.html`

Experimental Verification of Universality

- Universality isn't just a mathematical curiosity. Physical systems undergo period-doubling order-chaos transitions. Almost miraculously, these systems also appear to have a universal δ.

- Experiments have been done on fluids, circuits, acoustics:
  - Water: $4.3 \pm .8$
  - Mercury: $4.4 \pm .1$
  - Diode: $4.5 \pm .6$
  - Transistor: $4.5 \pm .3$
  - Helium: $4.8 \pm .6$

  Data from Cvitanović, *Universality in Chaos*, World Scientific, 1989.

- A very simple equation, the logistic equation, has produced a quantitative prediction about complicated systems (e.g., fluid turbulence) that has been verified experimentally.

- Nature is somehow constrained.

Detour: A Little Bit More About Universality

- The order-disorder phase transition in the logistic map is not the only sort of phase transition that is universal.
- Second order (aka continuous) phase transitions are also universal.
- There are several different universality classes, each of which has different values for quantities analogous to δ.
- The symmetry of the order parameter and the dimensionality of the space of the system determine the universality class.
- The order parameter is a quantity which is zero on one side of the transition and non-zero on the other.

Chaos: Deterministic Source of Randomness

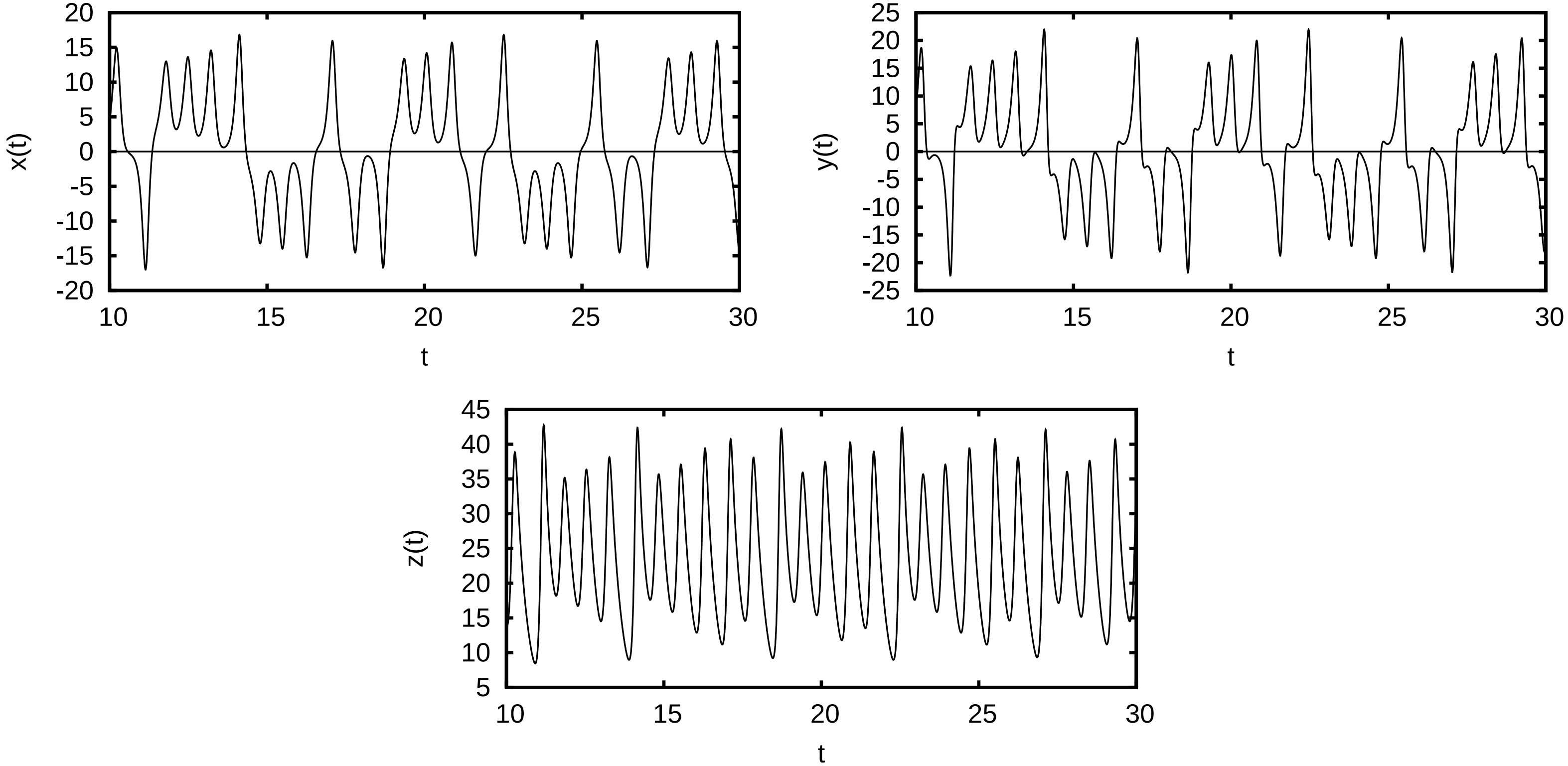
- A chaotic system behaves as if it is random, not governed by a deterministic rule.
- For $r = 4$, the symbolic dynamics of the logistic equation produce a sequence of 0's and 1's that is indistinguishable from a fair coin toss.
- Symbolic dynamics: 0 if $x < \frac{1}{2}$, 1 if $x > \frac{1}{2}$.
- The apparent randomness arises because the system is so deterministic. Determinism gives rise to SDIC.

Introduction to Chaos Part III: Strange Attractors

$$\begin{aligned} \frac{dx}{dt} &= \sigma(y-x) \\ \frac{dy}{dt} &= x(\rho-z)-y \\ \frac{dz}{dt} &= xy-\beta z \ . \end{aligned} \tag{1}$$

- The Lorenz Equations: introduced by Edward Lorenz in the early 1960's as a very simple model of a weather system.
- Here there are three variables, x, y, and z that change in time.
- The variables are continuous: defined for every time t, not just discrete times.
- The Lorenz equations are *differential equations*, a type of dynamical system where the rate of change at every instant is specified.
- From this rate of change, one can figure out how the variables themselves change.

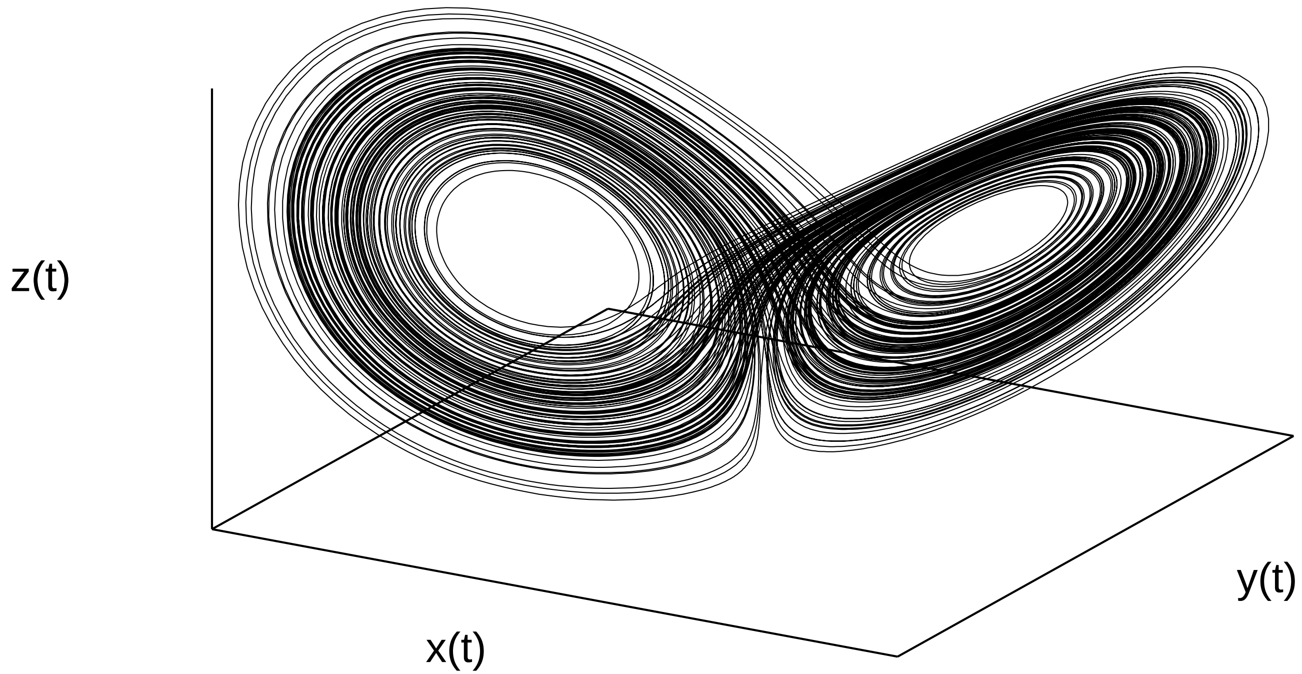
Lorenz Equations: Aperiodic Trajectories



- x, y,and z are all aperiodic. They do not repeat.
- How are x, y, and z related? To see this, let's plot the three variables on the same graph.

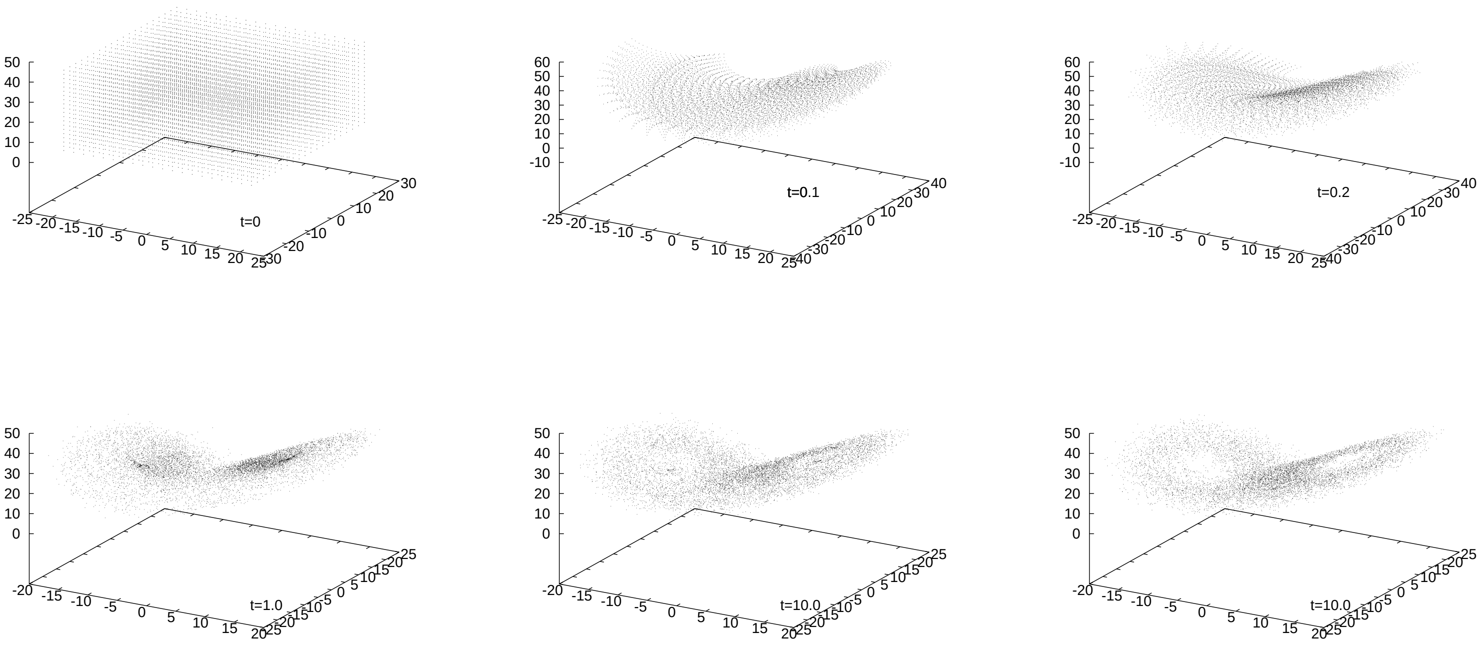
Parameter value: $\sigma = 10$ and $\beta = 2.667$, $\rho = 28$

Lorenz Equations: Strange Attractor



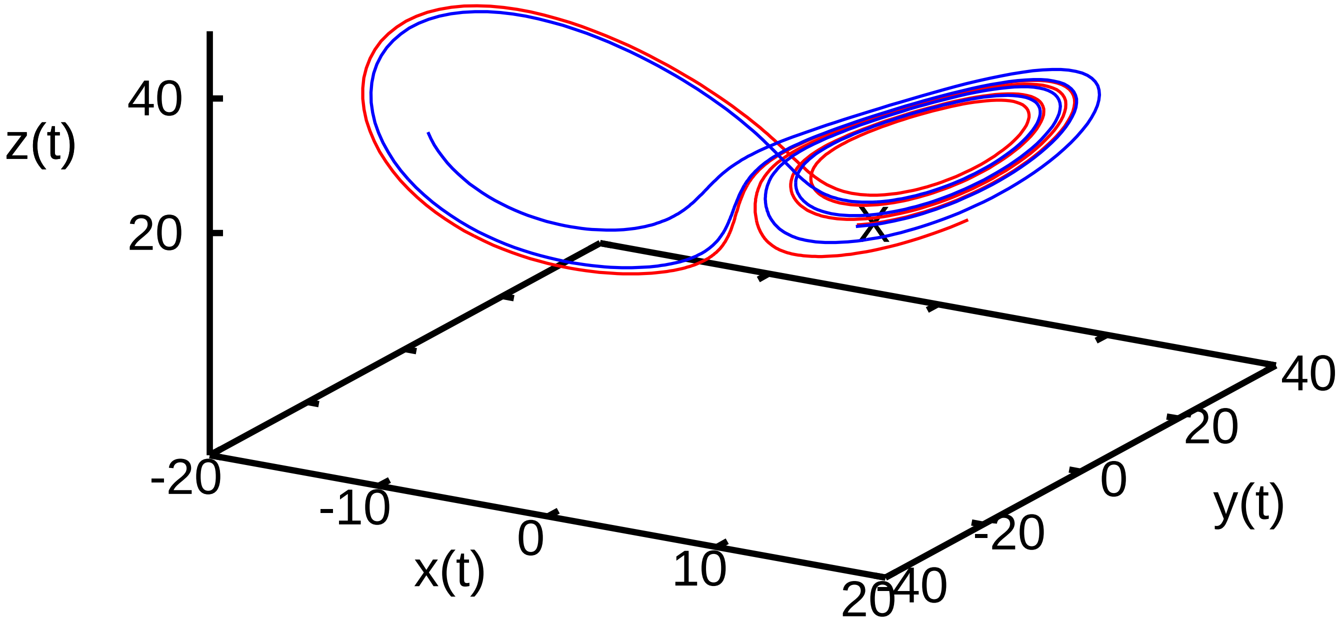
- Although individually x, y, and z move seemingly at random, when plotted together one can see a complicated relationship between them.
- The trajectory weaves through space but never repeats.

Lorenz Equations: Strange Attractor, continued



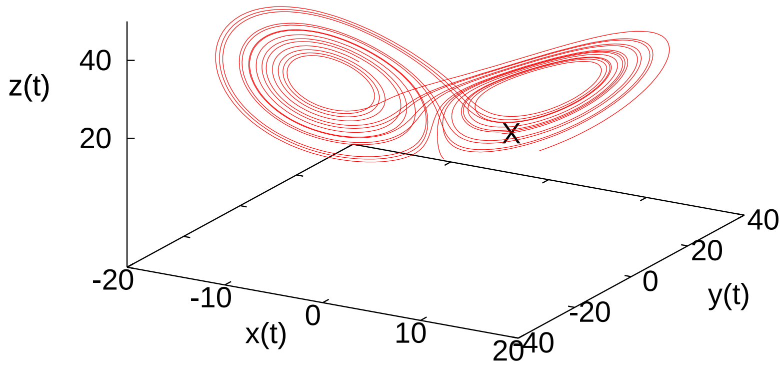
- This shape is an attractor. Orbits get pulled to it.
- Plot of 8000 different initial conditions uniformly distributed in a cube.
- The orbits are pulled to the attractor.

Lorenz Equations: Butterfly Effect



- The Lorenz Equations show the butterfly effect.
- The blue and red orbit start at almost exactly the same point, indicated by X.
- Very quickly the two orbits become quite different.

A Chaotic Attractor



- The attractor is stable; it attracts all orbits.
- But the dynamics on the attractor itself are chaotic.
- The system is a mix of order and unpredictability.
- Roughly speaking, unpredictability $\approx$ weather.
- Global structure, the shape of the attractor $\approx$ climate.
- Strange attractors are a sort of order hidden within chaos.

Chaos Conclusions

- Deterministic systems can produce random, unpredictable behavior. E.g., logistic equation with $r = 4$.
- Simple systems can produce complicated behavior. E.g., long periodic behavior in logistic equation.
- Some features of dynamical systems are universal—the same for many different systems.
- Chaos and other structures can be stable.
- Aubin and Dahan Dalmedico: [C]haos has definitely blurred a number of old epistemological boundaries and conceptual oppositions hitherto seemingly irreducible such as order/disorder, random/nonrandom, simple/complex, local/global, stable/unstable, ...

  Aubin and Dahan Dalmedico, *Historia Mathematica* 29 (2002), 167. doi:10.1006/hmat.2002.2351

Chaos $\Rightarrow$ Complex Systems

- Many researchers who did groundbreaking work in chaos in the 1970s and 1980s are now doing work in complex systems.
- Appreciation that complex behavior can have simple origins.
- Universality gives us some reason to believe that we can understand complicated and complex systems with simple models.
- More generally, order and disorder, simplicity and complexity, are seen to not be opposites or mutually exclusive categories.
- There is a surprising and delightful creativity to simple, iterated systems.
- Chaos and dynamical systems hint at how randomness, complexity, and structure may emerge out of a simple and deterministic(?) world.
- But it is just one thread in the complex systems tapestry.

Complex Systems Theory?

Is there a science or theory of complex systems? Can there be one? My hunch is that the answer is no, at least not in the usual sense of theory.

- Perhaps looking for a unifying theory of complex systems is to forget what's interesting about complex systems: that the whole is the greater than the sum of its parts, innovation and novelty occur, new things emerge.
- On the other hand, I don't think it's the case that every complex system is different. There may be some unifying tools, principles and ideas.
- My strong hunch is that a theory of complex systems will be primarily concerned with **methods** and **tools** as opposed to universal governing principles or equations.

What Good are Complex Systems?

- Complex systems gives us a rich set of tools and models that can be used to investigate a wide range of phenomena that are only partially understood via traditional methods.
- Complex systems provide a new set of paradigms or exemplars: e.g., logistic equation, random graphs, CAs, Schelling's tipping model, etc. These serve as stories we tell about what the world is like, and provide an important counterbalance to linear, reductive, "rational" models that still are predominant in many fields.
- The model systems of the sort I've focused on here may have little to say directly about complicated, real-world phenomena. However, these systems provide a very clear setting in which to explore the discovery of pattern, and fundamental tradeoffs between randomness and order. This can hone intuition when considering other, real-world complex systems.

What Good are Complex Systems?, continued

- I believe that there is an aesthetic and perhaps even normative component to the study of complex systems. Part of what the field has in common is a group of people with similar tastes and concerns and a sense of what is interesting:
  - How the world is put together, rather than how it's taken apart.
  - A fascination with patterns and their formation.
  - A fascination with diversity.
  - A willingness to take risks.
  - A recognition of interrelationships and complexity.

The End

Thanks for your comments and questions.