Local Complexity for Heterogeneous Spatial Systems

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Motivation

- What is structure/pattern/complexity?
- How can it be discovered and measured?
- How are information and memory shared across a spatial configuration?
- Do different sites play different roles?

Outline

1. Review of Entropy and Excess Entropy in One Dimension
2. Extensions to Two Dimensions
3. Extensions to Inhomogeneous Systems
4. Results for Spin Glasses

Entropy and Entropy Density

The Shannon Entropy H measures the uncertainty associated with a random variable:

$$H[S] \equiv \sum_s -\Pr(s) \log_2 \Pr(s) \ . \tag{1}$$

Consider a long, one-dimensional chain of variables:

$$\longleftarrow \ \boxed{}\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}\boxed{} \ \longrightarrow \ .$$

$H(L)$ is the entropy of an L-block:

$$H(L) \equiv H\Big[\overset{\longleftarrow L \longrightarrow}{\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}}\Big] \ . \tag{2}$$

Entropy density definition:

$$h_\mu \equiv \lim_{L \to \infty} \frac{H\Big[\overset{\longleftarrow L \longrightarrow}{\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}}\Big]}{L} \ . \tag{3}$$

Entropy Density

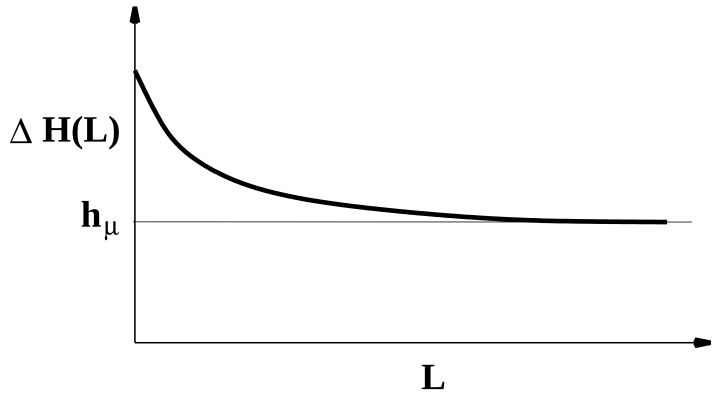
h_μ may also be written as a conditional entropy:

$$\begin{aligned} h_\mu(L) &= H(L) - H(L-1) && (4) \\ &= \Delta H(L) && (5) \\ &= H[\, \boxtimes \mid \overbrace{\boxtimes\square\square\square\square\square}^{(L-1)\longrightarrow} \,] \,. && (6) \end{aligned}$$

$$h_\mu = \lim_{L\to\infty} h_\mu(L) \qquad (7)$$

- h_μ is known as: **entropy rate**, **metric entropy**, and **entropy density**.
- h_μ is the irreducible randomness: the randomness that persists even after statistics over arbitrarily long sequences are taken into account

How does $\Delta H(L)$ approach h_μ?



- For finite L , $\Delta H(L) \geq h_\mu$. Thus, the system appears more random than it is.
- We can learn about the complexity of the system by looking at *how* the entropy density converges to h_μ.
- The **excess entropy** captures the nature of the convergence and is defined as the area between the two curves above:

$$\mathbf{E} \equiv \sum_{L=1}^{\infty} [\Delta H(L) - h_\mu] \ .$$

Excess Entropy

- $\mathbf{E}$ is thus the total amount of randomness that is “explained away” by considering larger blocks of variables.
- One can also show that $\mathbf{E}$ is equal to the mutual information between the “past” and the “future”:

$$\mathbf{E} = I(\overleftarrow{S}; \overrightarrow{S}) \equiv H[\overleftarrow{S}] - H[\overleftarrow{S} \mid \overrightarrow{S}] \; .$$

- $\mathbf{E}$ is thus the amount one half “remembers” about the other, the reduction in uncertainty about the future given knowledge of the past.
- Equivalently, $\mathbf{E}$ is the “cost of amnesia:” how much more random the future appears if all historical information is suddenly lost.
- The Excess Entropy is also known as the **Predictive Information** and the **Effective Measure Complexity**.

Excess Entropy and Entropy Rate Summary

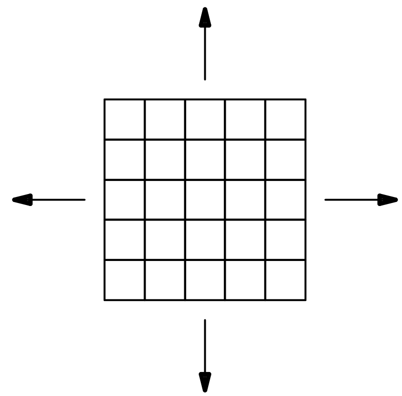
- Excess entropy $\mathbf{E}$ is a measure of complexity (order, pattern, regularity, correlation, structure ...)
- $\mathbf{E}$ detects periodic order of any periodicity $\mathbf{E} = \log_2(\mathrm{Period})$.
- Entropy rate h_μ is a measure of unpredictability.
- Both $\mathbf{E}$ and h_μ are well understood and have clear interpretations.
- Both $\mathbf{E}$ and h_μ are functions of the distribution over sequences.
- Both $\mathbf{E}$ and h_μ have been calculated for a wide range of systems.
- There are multiple forms for $\mathbf{E}$, all of which are the same in one dimension.

For more, see, e.g.,

- Crutchfield and Feldman, *Chaos*. **15**:23. 2003.
- D.P. Feldman, C.S. McTague, and J.P. Crutchfield. Chaos. 18:043106.

Entropy in Two Dimensions

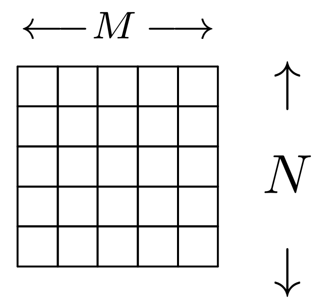
Consider a big, two-dimensional lattice of variables:



$H(M,N)$ is entropy of M,N-block:

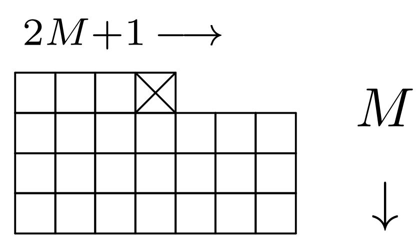
$$H(M,N) \equiv H\left[\begin{array}{cc} \overset{\longleftarrow M \longrightarrow}{\begin{array}{|c|c|c|c|c|}\hline \; & \; & \; & \; & \; \\ \hline \; & \; & \; & \; & \; \\ \hline \; & \; & \; & \; & \; \\ \hline \; & \; & \; & \; & \; \\ \hline \; & \; & \; & \; & \; \\ \hline \end{array}} & \begin{array}{c} \uparrow \\ N \\ \downarrow \end{array} \end{array} \right] . \qquad (8)$$

Two-Dimensional Entropy Convergence



$$h_\mu \equiv \lim_{M,N\to\infty} \frac{H\left[\quad\right]}{MN} . \qquad (9)$$

Is it possible to express the entropy density as the entropy of a single spin, conditioned on an appropriate semi-infinite block? **Yes.**



$$h_\mu(M) \equiv H\left[\ \boxtimes \mid \quad\right] . \qquad (10)$$

$$h_\mu = \lim_{M\to\infty} h_\mu(M) . \qquad (11)$$

Two-Dimensional Excess Entropy

- We can then define the excess entropy:

$$\mathbf{E}_c \equiv \sum_{M}^{\infty} \left(h_\mu(M) - h_\mu \right) \ . \tag{12}$$

- Note: One can also define a form $\mathbf{E}_i$ for the two-dimensional excess entropy based on the mutual information.
- $\mathbf{E}_i \neq \mathbf{E}_c$, but the two behave similarly.
- As in 1D, $\mathbf{E}$ distinguishes among periodic patterns of all periods.
- For more on 2D Excess Entropy, see D.P. Feldman and J.P. Crutchfield, *Phys. Rev. E.* **67**:051104. 2003.

Regular Ising Model (Not a Spin Glass)

- Spins $s_i \in \{-1, +1\}$ interact with their nearest neighbors.

$$\text{Energy of state} = -J \sum_{<ij>} s_i s_j \ , \tag{13}$$

  where the sum runs over nearest neighbors. Then,

$$\text{Probability of state } i \propto e^{-E_i/T} \tag{14}$$

  where $T =$ temperature.

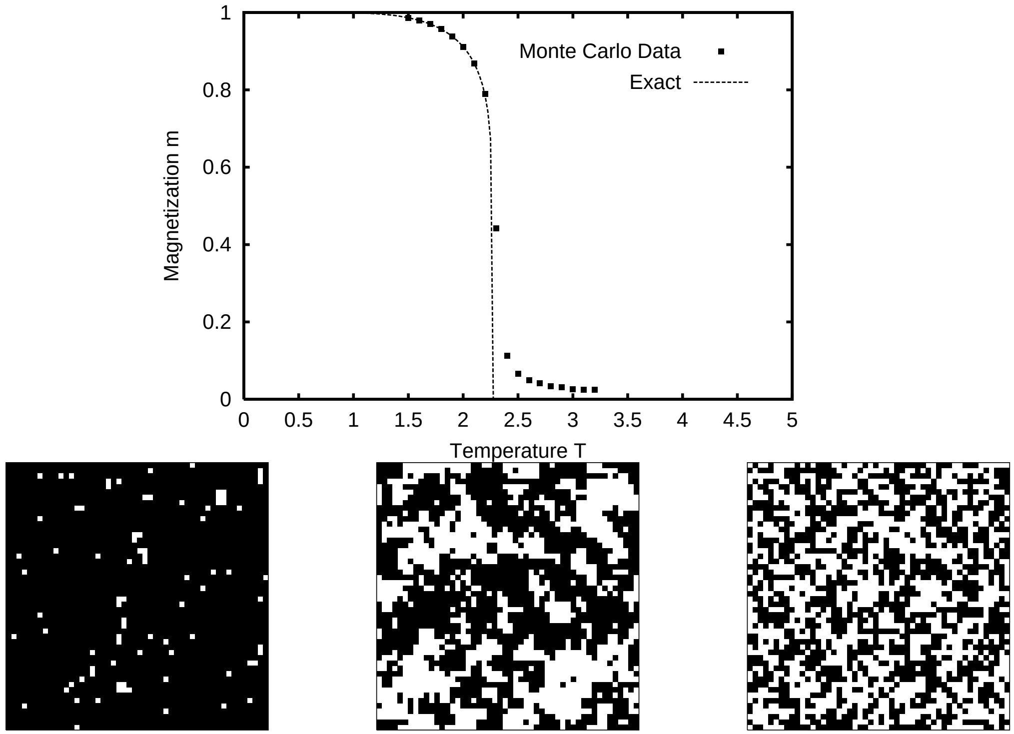
- As the temperature is lowered, the system orders. The magnetization m acquires a non-zero expectation value.

- The magnetization is defined by

$$m = \langle \sum_i s_i \rangle$$

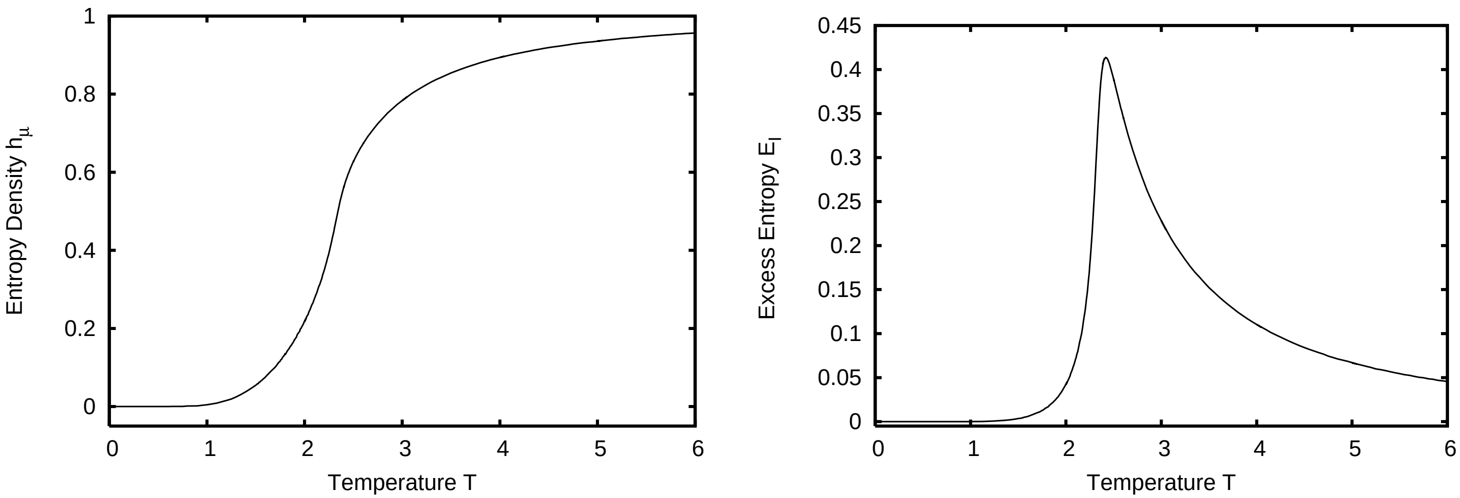
  where the angular brackets indicate averaging over states using Eq. (14).

Ising Model Phase Transition



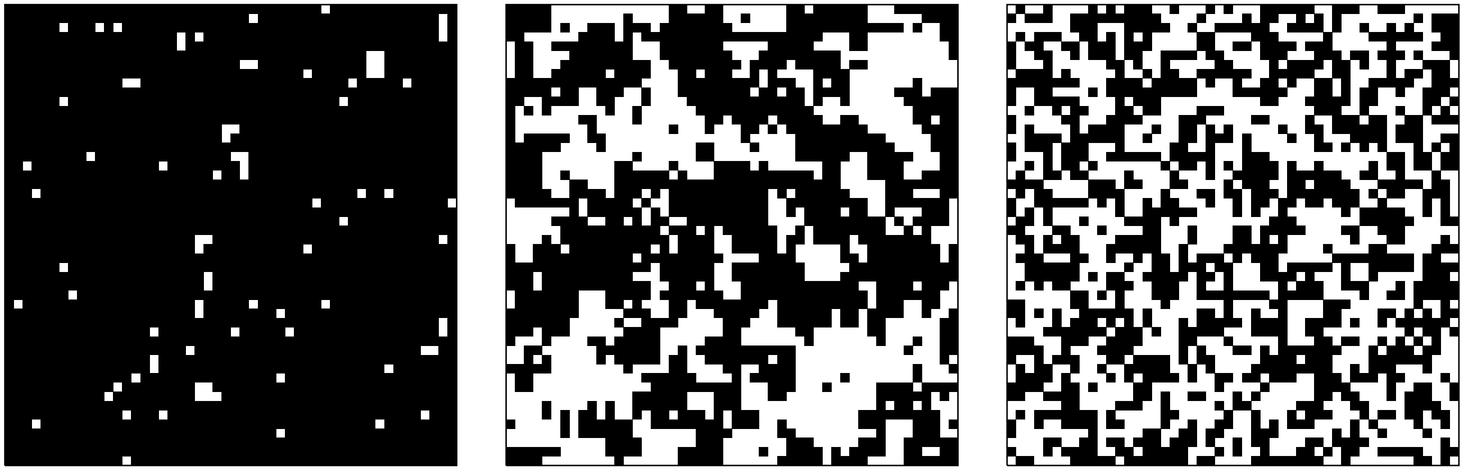
- Below the critical temperature, the symmetry is broken – up and down are no longer equally likely.
- Below T_c the system's ergodicity is broken. It no longer can get to any state from any other state in finite time.

2D Ising Model Phase Transition



- Convergence form of the excess entropy $\mathbf{E}_{\rm c}$ vs. entropy density h_μ versus temperature T for the two-dimensional Ising model with NN couplings and no external field.
- Model undergoes phase transition as T is varied at $T \approx 2.269$.
- There is a peak in the excess entropy near the transition temperature.
- Results via Monte Carlo simulation of 100x100 lattice.

Ising Model Configurations



- Typical configurations for the 2D Ising model below, at, and above the critical temperature.

Spin Glasses

- Spin glasses are usually disordered and frustrated. First, disorder:
- Spins $s_i \in \{-1, +1\}$ interact with their nearest neighbors. But this time, the interaction is random

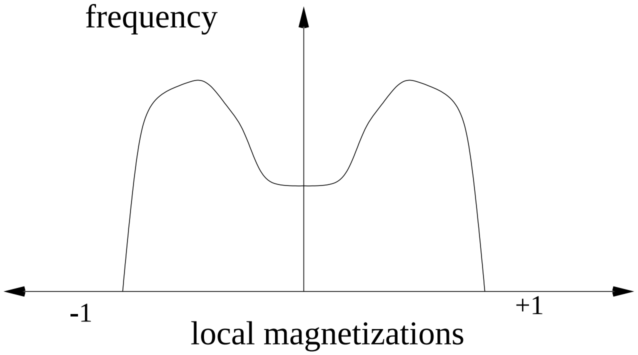
$$\text{Energy of state} = -\sum_{<ij>} J_{ij} s_i s_j \ , \tag{15}$$

where the sum runs over nearest neighbors, and J_{ij} is a random variable. Usually, $J_{ij} = +1$ with probability $1/2$.

- Spin glasses are used to model a wide range of situations in which entities interact through random interactions and/or which exhibit frustration.
- What happens as the system is cooled? The ground state is disordered – the interactions, on average, make the spins neither align nor anti-align.

Typical ground state of spin glass.

- The magnetization m is zero for all temperatures.
- However, the local magnetizations $m_i = \langle s_i \rangle_t$ are non-zero
- Some spins get frozen mostly up or down.

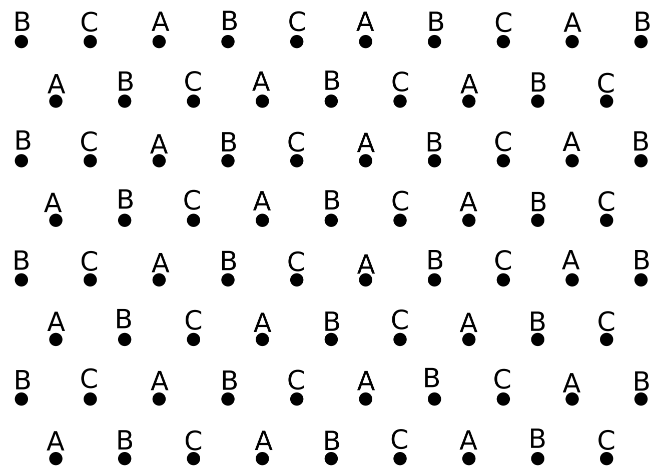


- Figure from `http://www.informatik.uni-koeln.de/ls_juenger/`.

Features of spin glasses

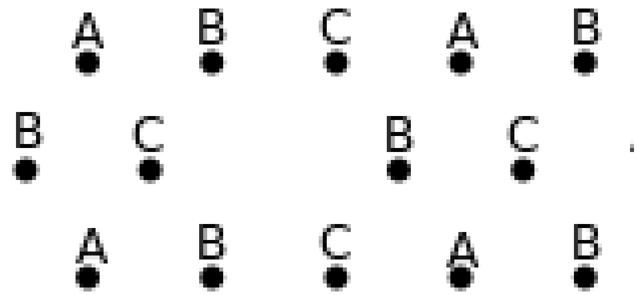
- Multiple ground states, not related by simple symmetry operation.
- Loss of ergodicity. Ergodic components aren't related by any simple symmetry.
- Low temperature state is "frozen" and "disordered."
- **Local averages are not the same as global averages.**
- **The degree to which the local averages differ is often used to quantify the degree of glassiness:**

Frustration



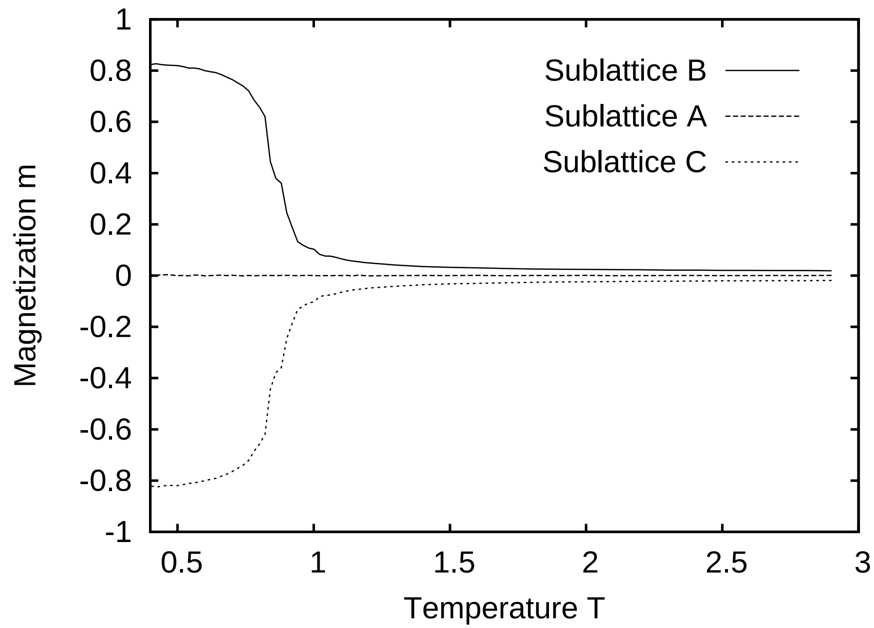
- Suppose all interactions are anti-ferromagnetic: all spins want to anti-align.
- **Frustration:** All energetic constraints can't be satisfied at the same time.
- Note that the triangular lattice naturally decomposes into three sublattices.
- **How might we relieve frustration?**

Dilution: The Kaya-Berker Model



- Dilute the system by deleting spins on one of the sublattices.
- This relieves frustration.
- If the dilution is sufficiently large, the two non-diluted sublattices order, and the diluted sublattice undergoes a spin-glass transition.
- Kaya and Berker, PRE, **62**, R1469. (2000).
- This is an example of **order by disorder**: adding disorder in the form of random deletions causes the system to order.

Magnetization for the Kaya-Berker Model

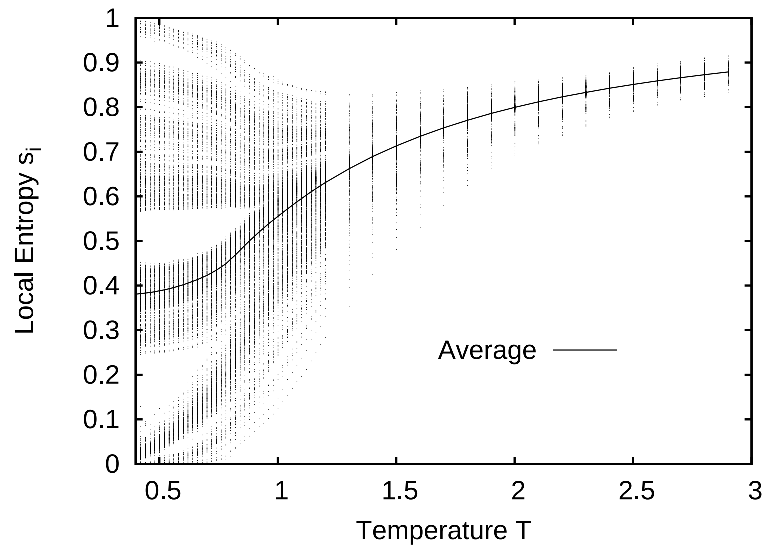


- There is a net magnetization on lattices B and C.
- For this and all results that follow, sublattice A is 15% diluted.
- Monte Carlo results, 99×99 lattice.

Local or Global Entropies?

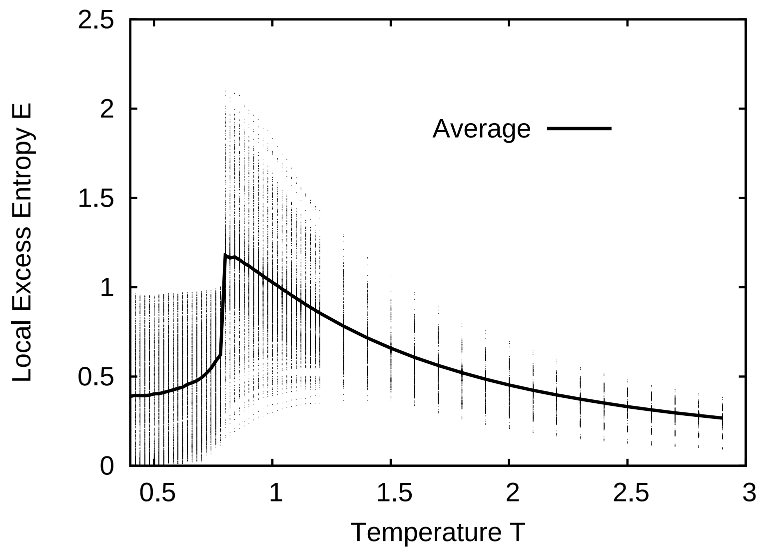
- As with the magnetization, when calculating h_μ and $\mathbf{E}$ we can do the averaging in two ways.
  1. Average over all blocks at all locations.
  2. Average over a single block pinned at a particular site.
- Method 2 gives one a local entropy: a measure of the unpredictability at a particular site.
- The local entropies are not the same as the global entropy density in the glassy state.
- Robinson, Feldman, McKay. *Chaos*. **23**(3):037114. 2011.

Local Entropies



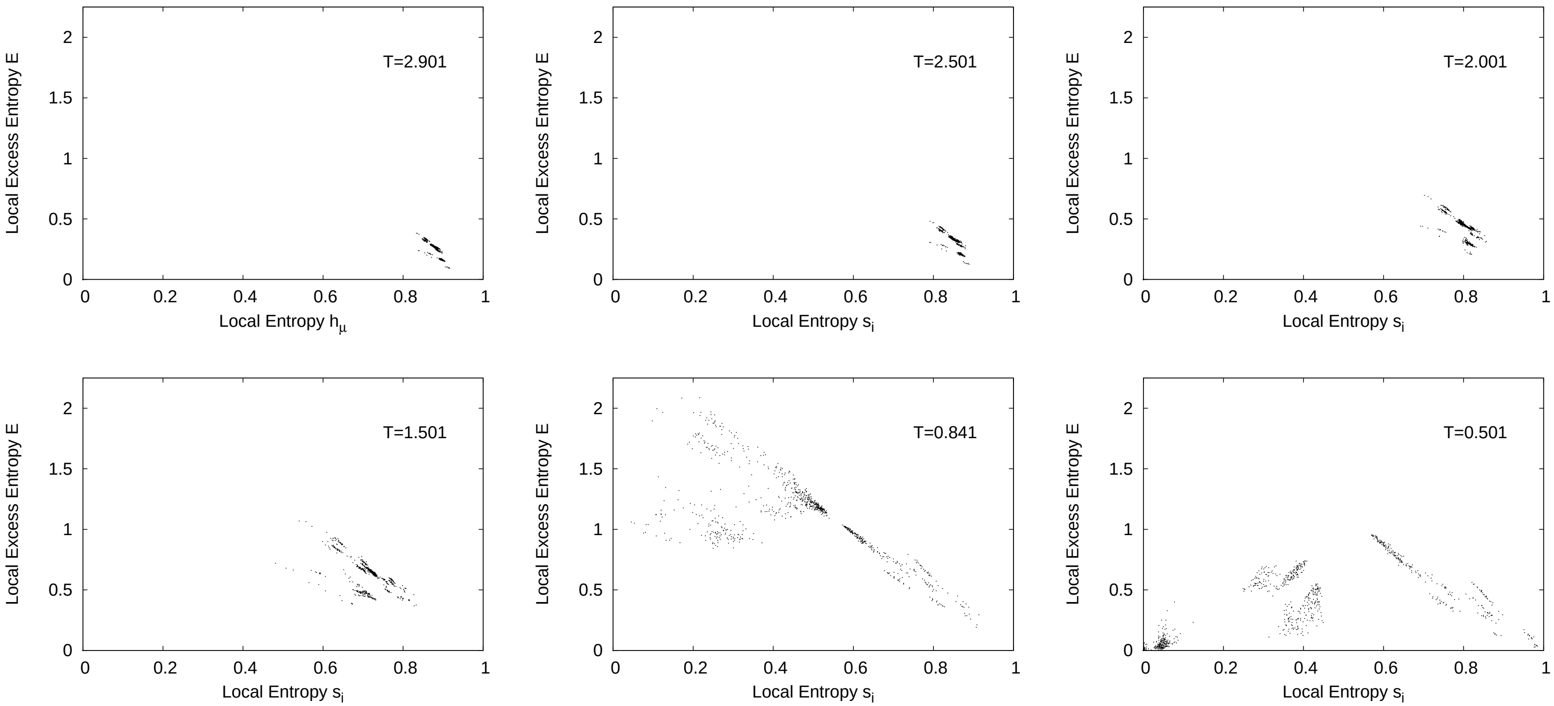
- Information is not shared equally across the lattice.
- The average of the local entropies equals the global, thermodynamic entropy.
- The global entropy can be calculated information theoretically or by integrating the thermodynamic relation $ds = (c/T)dT$.
- Note that on some sites the local entropy increases as temperature is lowered.

Local Excess Entropies



- Memory is not shared equally across the lattice.
- Note the sharp drop at $T \approx 0.8$. This coincides with the critical temperature, measured by other means.
- **E** is a general-purpose order parameter.

Complexity-Entropy Diagrams for the Spin Glass



Summary

- Excess entropy and entropy rate are well understood measures of structure and unpredictability in 1D.
- Excess entropy can be extended to 2D, although the extension is not unique.
- Entropy of a disordered lattice system can be decomposed into local contributions to see how entropy is distributed across the lattice.
- A local excess entropy can similarly be calculated. It shows how memory or structure is distributed across the lattice.
- Different sites play different roles.
- Average excess entropy is highly sensitive to structural changes in the configurations.

Frontiers: Promising Areas for Future Work

- Local information and complexity measures for distributed systems.
- Networks: dynamics on networks and the structure of the networks themselves.
- Empirical work, “big data.”
- Closer ties to thermodynamics
- Closer ties to statistics and statistical inference

The End

- Thanks for your attention and questions.