The Definite Integral

Our task was to determine the total distance the cat ran. We wanted to use

$$d = v\Delta t \ , \tag{1}$$

but we couldn't, because v was changing over the time interval Δt. So we just decided to pretend that v wasn't changing and use Eq. (1) anyway.

But immediately, if not sooner, there was a general sense that this was pretty lame. This fiction wasn't going to give us a good estimate of how far the cat really went. So, we decided to divide the time interval into two smaller intervals. We then again pretended that the cat's speed was constant over each of these intervals. The cat's speed isn't constant, but it's closer to constant, because the time interval is smaller.

We then realized, again quite quickly, that we could do even better if we divided the total time interval into smaller and smaller time intervals. We referred to these approximations as left and right hand sums:

$$\text{Left hand sum} = v(t_0)\Delta t + v(t_1)\Delta t + v(t_2)\Delta t + \cdots + v(t_{n-1})\Delta t \ . \tag{2}$$

$$\text{Right hand sum} = v(t_1)\Delta t + v(t_2)\Delta t + v(t_3)\Delta t + \cdots + v(t_n)\Delta t \ . \tag{3}$$

Sigma Notation. The symbol "Σ", a capital sigma, is used to denote summation. For example,

$$\sum_{n=1}^{4} n^2 = 1^2 + 2^2 + 3^2 + 4^2 \ . \tag{4}$$

We can use sigmas to write the left and right hand sums more compactly:

$$\text{Left hand sum} = \sum_{i=0}^{n-1} v(t_i)\Delta t \tag{5}$$

and

$$\text{Right hand sum} = \sum_{i=1}^{n} v(t_i)\Delta t \tag{6}$$

We argued in class last time that the total change in position only becomes exact as we take more and more subintervals. In the above two equations, this corresponds to taking the $n \to \infty$ limit.

This business of summing up a function over smaller and smaller intervals goes by a special name: **the definite integral**. The definite integral of f from a to b is written:

$$\int_a^b f(t)\,dt\ . \tag{7}$$

The definite integral is defined to be the $n \to \infty$ limit of the left or right hand sums of subdivisions of $[a, b]$, where n is the number of subdivisions.

$$\int_a^b f(t)\,dt \equiv \lim_{n\to\infty} \left(\sum_{i=0}^{n-1} f(t_i)\,\Delta t \right)\ . \tag{8}$$

or

$$\int_a^b f(t)\,dt \equiv \lim_{n\to\infty} \left(\sum_{i=1}^{n} f(t_i)\,\Delta t \right)\ . \tag{9}$$

Note that

- The notation is Eq. (7) is shorthand for an entire *process*. The definite integral represents the result of using smaller and smaller Δt's in left- or right-hand sums.
- In this sense, the definite integral is like the derivative: $f'(x)$ is also shorthand for an entire process.
- The definite integral is a number, not a function.
- The definite integral of Eq. (7) may also be interpreted as the area between the curve $f(x)$ and the x-axis.