Chapter 10.2 More with Taylor Series

Calculus II

Spring 2021

College of the Atlantic

In a previous episode of Calculus II, we considered approximating a function $f(x)$ by a polynomial $P(x)$:

$$f(x) \approx P(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots \,, \tag{1}$$

where

$$C_n = \frac{f^{(n)}}{n!} \,. \tag{2}$$

We derived this equation by requiring that all the derivatives of $f(x)$ and $P(x)$ agree at $x = 0$.

Famous and Useful Taylor Series

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \,. \tag{3}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots \,. \tag{4}$$

$$\sin(x) = ?? \tag{5}$$

1. Determine the Taylor series for $\sin(x)$ using Eq. (2).
2. Once you have the Taylor series for $\sin(x)$, take its derivative. Does the answer look familiar?
3. Take the derivative of the Taylor series for e^x. What do you get?
4. What is
$$\lim_{x \to \infty} \frac{\sin(x)}{x} \,? \tag{6}$$
To answer this question, express $\sin(x)$ as a Taylor series.
5. Find the Taylor series for $f(x) = e^{-x^2}$. There is an easy and a hard way to do this. Choose the easy way.
6. Use your answer to the previous question to find a series representation of the anti-derivative of e^{-x^2}.

The Binomial Series

1. Use Eq. (2) to determine the first three terms in the Taylor Series of $f(x) = (1+x)^p$ near $x = 0$. This Taylor series is called the *binomial series*.

2. Use the binomial series to determine approximate numerical values for:

$$\frac{1}{0.99^2} \tag{7}$$

$$\frac{1}{0.98} \tag{8}$$

$$\frac{1}{0.999^2} \tag{9}$$