Class 28: Using Taylor Series
Calculus II
College of the Atlantic. March 13, 2023

In a previous episode of Calculus II, we considered approximating a function $f(x)$ by a polynomial $P(x)$:

$$f(x) \approx P(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots \,, \tag{1}$$

where

$$C_n = \frac{f^{(n)}}{n!} \,. \tag{2}$$

We derived this equation by requiring that all the derivatives of $f(x)$ and $P(x)$ agree at $x = 0$.

Super Famous and Super Useful Taylor Series

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \,. \tag{3}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots \,. \tag{4}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} - \cdots \,. \tag{5}$$

1. Take the derivative of the Taylor series for $\sin(x)$. Does the answer look familiar?

2. Take the derivative of the Taylor series for e^x. What happens?

3. What is
$$\lim_{x \to \infty} \frac{\sin(x)}{x} \,? \tag{6}$$
To answer this question, express $\sin(x)$ as a Taylor series.

4. Find the Taylor series for $f(x) = e^{-x^2}$. There is an easy and a hard way to do this. Choose the easy way.

5. Use your answer to the previous question to find a series representation of the anti-derivative of e^{-x^2}.

The Binomial Series: Also Quite Famous and Definitely Super Useful

1. Use Eq. (2) to determine the first three terms in the Taylor Series of $f(x) = (1+x)^p$ near $x = 0$. This Taylor series is called the *binomial series.*

2. Use the binomial series to determine approximate numerical values for:

$$\frac{1}{0.99^2} \tag{7}$$

$$\frac{1}{0.98} \tag{8}$$

$$\frac{1}{0.999^2} \tag{9}$$

3. In special relativity, the time interval $\Delta\tau$ experienced by someone moving at a constant speed v is given by

$$\Delta\tau = \sqrt{1-(v/c)^2}\Delta t\ , \tag{10}$$

   where Δt is the time interval as measured in an inertial ("at rest") reference frame, and c is the speed of light, which is around 3×10^8 m/s.

   (a) Use a first-order binomial series approximation for the square root in Eq. (10) to derive an expression for $\Delta\tau - \Delta t$.

   (b) Suppose a clock takes a plane flight from Boston to San Francisco. The flight takes six hours ($21,600s$) according to clocks on the ground. The plane flies at a speed of 230 m/s. How much longer does the flight take as measured by the clock on the plane?

Hey. Remember Imaginary Numbers?

1. Write down the Taylor series for $f(x) = e^{ix}$. Collect real and imaginary terms. What do you notice?