Class 02: Accumulated Change

Calculus II

College of the Atlantic. January 8, 2025

Figure 1: Unicorn mosaic on a 1213 church floor in Ravenna, Italy. Public domain image uploaded by Watchduck to wikipedia. Image source `https://en.wikipedia.org/wiki/Unicorn#/media/File:San_Giovanni_Evangelista_in_Ravenna,_unicorn.jpg`.

Last time we tried to figure out how far a unicorn ran, given knowledge of its speed. This was easy if the unicorn was running at a constant speed.

> *At dinnertime, whoever is cooking dinner rings the dinner bell when food is ready. A unicorn hears the bell, and after a short delay runs to get dinner. The unicorn runs at a velocity of* 10 *m/s for* 4 *seconds. How far has the unicorn run?*

But if the unicorn isn't running at a constant velocity, the situations is harder. The reason for this is that the formula $d = vt$ is true only if the velocity is constant.

For example, consider this situation:

> *A different unicorn hears the bell, waits, and then starts running quickly and then slows down. The unicorn's velocity is given by $v(t)$, where v is measured in m/s, and t is measured in seconds since the bell rang. Values for $v(t)$ are shown in the table below:*

t	$v(t)$
2	20
4	15
6	10
8	5
10	0

To deal with this, we had to take the full time interval from 2 to 10 seconds and break it up into 4 sub-intervals each two seconds long. We can denote the durations of the sub-intervals by Δt.

For each interval, we pretend that the unicorn velocity is constant. For example, for the first time interval we might pretend that the velocity was 20 m/s for the entire two seconds, and then would find a distance traveled of $d = v\Delta t = (20)(2) = 40$m. We did similar things for the remaining three intervals, added up the distances for each interval and got an estimate for the total distance traveled.

This is an estimate, since the velocity isn't actually constant over each time interval, but we are pretending that it is. We will get an over- or under-estimate[1]of the exact distance traveled depending on if we chose the larger or smaller velocity value for each interval.

Some terminology. If one consistently chooses the first value when doing an approximation like this, the resulting total distance is called the *left-hand sum*. If you choose the second value, you've calculated a *right-hand sum*.

The leads to the central **big question** for the course: How do calculate total distance traveled if the velocity isn't constant? And under what conditions can we determine this exactly[2]? We will spend the first few weeks of the class working on this.

Btw, there is another **almost-as-big question**[3] lurking: What might *average velocity* mean in a situation where the velocity is changing all the time? How could we calculate this average. On the one hand, such an average seems like a straightforward enough concept. But its calculation presumably isn't as simple as just taking a few velocities, adding them up, and dividing.

Also btw, we noticed a short-cut way of figuring out the difference between the upper and lower estimates. Remember what that was?

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[1] An important detail: This is only true if the velocity is only decreasing or only increasing. It's not true if the velocity goes up and down.

[2] And what does "exactly" mean, exactly?

[3] Actually, this is the same big question. It's not really different.

Here is the last problem we did in class on Monday.

> *Yet another unicorn hears the dinner bell and runs to dinner. How far as it run? Its velocity is given by the $v(t)$ values shown below.*

t	$v(t)$
2	40
4	23
6	10
8	2.5
10	0

We decided that in order to get a more accurate estimate of the distance this unicorn has run, we will need more data. Here is the data we need:

t	$v(t)$
2	40
3	30.63
4	22.5
5	15.63
6	10
7	5.63
8	2.5
9	0.63
10	0

1. Calculate upper and lower estimates for the total distance ran by the unicorn.

2. What δt would you need so that the difference between the upper and lower estimates was no larger than 0.1?