14.4: Directional Derivatives and the Gradient Vector

Calculus III

College of the Atlantic

1. Let $f(x, y) = 3x^2y$.

   (a) Find the gradient of f at $x = 1, y = 2$.

   (b) Find the directional derivative $f_{\vec{u}}$ at $(1, 2)$ for the following $\vec{u}$'s:

      i. $\vec{u} = \vec{i}$
      ii. $\vec{u} = \vec{j}$
      iii. $\vec{u} = \vec{i} + \vec{j}$
      iv. $\vec{u} = \vec{i} - \vec{j}$
      v. $\vec{u} = -\vec{i}$
      vi. $\vec{u} = -\vec{j}$

   (c) In what direction $\vec{u}$ is $f_{\vec{u}}$ the largest?

   (d) In what direction $\vec{u}$ is $f_{\vec{u}}$ the smallest?

   (e) In what direction $\vec{u}$ is $f_{\vec{u}}$ the zero?

2. Consider the function $f(x, y) = x^2 + 4y^2$.

   (a) Sketch contour lines for the function in the first quadrant.

   (b) Calculate the gradient vector for general x, y.

   (c) Determine the value of the gradient vector at the following points:

      i. $(1, 1)$
      ii. $(1, 2)$
      iii. $(2, 1)$
      iv. $(2, 2)$

   (d) Draw the above gradient vectors on your contour plot sketch. Do the values make sense geometrically?

   (e) What is the rate of change of f at $(2, 2)$ in the direction $\vec{u} = -\vec{i} + 2\vec{j}$?

   (f) In what direction is the rate of change of f at $(2, 2)$ the largest? I.e., in what direction is the function the steepest uphill?

   (g) In what direction is the rate of change of f at $(2, 2)$ the smallest? I.e., in what direction is the function the steepest downhill?

   (h) In what direction is the rate of change of f at $(2, 2)$ zero? I.e., in what direction does the function not change?

3. A caterpillar is on a metal surface whose temperature is given by $T(x, y) = 3x^2y - y^3$. The caterpillar does not like heat. It is at the point $(5, 1)$.

   (a) In what direction should it move so that it gets cooler as quickly as possible?

   (b) If it initially moves at 0.8 cm/s, at what rate does the caterpillar experience a temperature decrease?

4. A bird is flying through a large cloud of pollution whose distribution is given by $\rho(x, y, z) = xz + 3x^2y - y^3$ in units of grams per cubic meter, where x, y, and z are measured in miles. The bird does not like pollution. It is at the point $(1, 2, 1)$.

   (a) In what direction should it move so that it gets to cleaner air as quickly as possible?

   (b) What are the units of the gradient vector?

   (c) If it initially flies at 1.2 m/s, at what rate does the bird experience a pollution decrease?

5. Consider the function $f(x, y, z) = e^{-(x^2+y^2+z^2)}$.

   (a) Calculate $\vec{\nabla} f$.

   (b) Determine the gradient vector at the following points

      i. $(0, 0, 0)$
      ii. $(1, 0, 0)$
      iii. $(0, 0, 1)$
      iv. $(1, 1, 1)$

   (c) What is the gradient vector at the origin? What does your answer mean?

   (d) What is the directional derivative in the $-\hat{z}$ direction at the point $(1, 0, 0)$.

   (e) What is the directional derivative in the $-\hat{z}$ direction at the point $(0, 0, 1)$.