14.5–6: Gradient Vectors, Chain Rules

Calculus III

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1. Let the altitude of a surface be given by $z(x, y) = 2 + 3x^2 + 4y^3$, where x, y, and z are measured in kilometers.
   (a) Find $\vec{\nabla} f(2, 1)$.
   (b) Suppose you are at the point (2, 1) and are moving at 2 km an hour northwest. At what rate are you gaining or losing altitude?
   (c) In what direction should you head so that you are gaining altitude most quickly?
   (d) In what direction should you head so that your altitude is constant?

2. For some unknown reason, a square room is slowly expanding. All of its walls are increasing at a rate of 0.2 meters/day. How fast is the area of the room increasing when the side of the room is 8 meters long?

3. Let $f(a, b) = a^2b^3$. At a particular moment in time, $a = 3$ and $b = 4$. At this moment, a is increasing at a rate of 2 units per second, while b is decreasing at 3 units per second. How fast is the function changing at this moment?

4. Suppose that z is a function of x and y: $z = f(x, y)$. And suppose that x and y are both functions of u and v: $x = g(u, v)$ and $y = h(u, v)$. How does z vary with u? To answer this question you will need to derive a new chain rule formula.

14.7: Second Derivatives

Calculus III

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5. Consider $g(x) = \sin(x^3y^4)$. Calculate g_{xx} and g_{xy}.
6. Let the temperature along a metal rod be given by $H(x,t)$, where: H is measured in Celsius degrees; x, the distance from the left end of the rod, is measured in centimeters; and t in minutes. Interpret the following equations:
   (a) $H(50,3) = 123$.
   (b) $H_t(50,3) = -2$.
   (c) $H_x(50,3) = -0.2$.
   (d) $H_{tx}(50,3) = 0.05$.
7. Let the temperature in a metal rod be given by the function $T(x,t) = 100e^{-t}\sin(\pi x)$, where t is measured in minutes and x in meters. The rod is one meter long. (So $0 \leq x \leq 1$.)
   (a) Sketch $T(x,0)$ and $T(x,0.1)$.
   (b) Using the two sketches you just drew, determine the signs of f_x, f_t, f_{xx}, and f_{xt} at $x = 0.2$.
   (c) Using the two sketches you just drew, determine the signs of f_x, f_t, f_{xx}, and f_{xt} at $x = 0.5$.
   (d) Using the two sketches you just drew, determine the signs of f_x, f_t, f_{xx}, and f_{xt} at $x = 0.8$.