~~All~~ Most of Calculus

Calculus III

College of the Atlantic. Winter 2022

0. The idea of a limit. We say $\lim_{x\to a} f(x) = c$ if $f(x)$ gets closer and closer to c as x gets closer and closer to a.

1. Definition and Interpretations of The Derivative

$$
\begin{aligned}
f'(x) &= \frac{df}{dx} && (1)\\
&= \text{slope of line tangent to } f(x) && (2)\\
&= \text{instantaneous rate of change of } f(x) && (3)\\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} && (4)
\end{aligned}
$$

2. Derivative Shortcuts
   (a) Power rule, trig functions, exponentials, logarithms
   (b) Chain rule
   (c) Product rule, ~~quotient rule~~

3. Derivative Applications
   (a) Critical points and optimization
   (b) Understanding functions and their rates of change
   (c) Geometric understanding of functions

4. Definition and Interpretations of the Integral
   (a) Definition of definite as limit of left- and right-hand sums, Riemann sums.
   (b) Definite integral as area under a curve.
   (c) Definite integral as sum of small changes.

5. Fundamental Theorem of Calculus
   (a) First fundamental theorem: total change of function equals sum of little changes:

$$\int_a^b f(x)\,dx = F(b) - F(a)\ , \ \text{where } F'(x) = f(x)\ . \qquad (5)$$

   (b) Constructing antiderivatives graphically and numerically
   (c) Second fundamental theorem:

$$\frac{d}{dx}\int_a^x f(z)\,dz = f(x) \qquad (6)$$

6. Integration Techniques
    (a) Guess and check
    (b) u substitutions
    (c) Integration by parts, ~~trig substitutions~~, ~~partial fractions~~
    (d) Computers and integral tables
    (e) Improper integrals
7. Integration Applications
    (a) Areas and volumes, volumes of revolution
    (b) Present and future values
    (c) Probability density and cumulative density functions
8. Sequences and series
    (a) Convergence of sequences
    (b) Geometric series
    (c) Convergence tests: comparison, limit comparison, alternating, ratio
    (d) Power series
    (e) Taylor series
9. Multivariable Differential Calculus
    (a) Partial derivatives, $\frac{\partial f}{\partial x}$: rate of change of $f(x, y)$ in x-direction.
    (b) Directional derivative, $f_{\vec{u}}(x, y)$: rate of change of $f(x, y)$ in $\vec{u}$-direction.
    (c) The gradient vector ∇f:
    $$\nabla f(x, y) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} \tag{7}$$
        i. ∇f points in direction of greatest change of f
        ii. $||\nabla f||$ slope in this direction of greatest change
        iii. $f_{\vec{u}} = \nabla f \cdot \vec{u}$
    (d) Optimization, Lagrange multipliers
10. Multivariable Integral Calculus
    (a) Iterated integrals
    (b) Integrals as volumes under surfaces
    (c) Multiple integrals in polar, cylindrical, and spherical coordinates

11. Vector Calculus: Basic Definitions

   (a) Vector fields: $\vec{F} \;=\; F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$
   (b) Line integrals: $\int_C \vec{F} \cdot \vec{dr}$, net "push" of $\vec{F}$ along curve C.
   (c) Flux integrals: $\int_S \vec{F} \cdot \vec{dA}$, flow across the surface S.
   (d) Divergence of a vector field:

$$\text{div}\vec{F} \;=\; \nabla \cdot \vec{F} \;=\; \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}. \tag{8}$$

   (e) Divergence as flux density of a field.
   (f) Curl of a vector field:

$$\text{curl}\vec{F} \;=\; \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k} \tag{9}$$

   (g) Curl as circulation density, "twistiness" of a field

12. Curl-less Fields. The following statements are equivalent

   (a) $\int_C \vec{F} \cdot \vec{dr} = 0$ for all closed curves C
   (b) $\int_C \vec{F} \cdot \vec{dr}$ is independent of path.
   (c) $\nabla \times \vec{F} = 0$, the field $\vec{F}$ is irrotational
   (d) There is a scalar function f such that $\nabla f = \vec{F}$. We call f a potential function for $\vec{F}$.

13. Divergence-less Fields. The following statements are equivalent

   (a) $\oint_X \vec{F} \cdot \vec{dA} = 0$ for all closed surfaces S
   (b) $\int_S \vec{F} \cdot \vec{dA}$ is independent of surface.
   (c) $\nabla \cdot \vec{F} = 0$.
   (d) There is a vector field $\vec{A}$ such that $\nabla \times \vec{A} = \vec{F}$. We call $\vec{A}$ a vector potential for $\vec{F}$.

14. Fundamental Theorem of Vector Fields: any field $\vec{F}$ can be written as the sum of a divergence-less and a curl-less vector field.

15. Fundamental Theorems for Vector Calculus

$$\int_C \nabla f \cdot \vec{df} \;=\; f(B) - f(A) \tag{10}$$

$$\int_S (\nabla \times \vec{F}) \cdot \vec{dA} \;=\; \int_{\partial S} \vec{F} \cdot \vec{dr} \tag{11}$$

$$\text{Total curl flux through area} \;=\; \text{Circulation around boundary}\ . \tag{12}$$

$$\int_V (\nabla \cdot \vec{F})\, dV \;=\; \int_{\partial V} \vec{F} \cdot \vec{dA} \tag{13}$$

$$\text{Total divergence in volume} \;=\; \text{Total flux out of surface}\ . \tag{14}$$

16. Another way to think about the material from Calculus III:

(a) Objects we worked with

i. Scalar function $f(x, y, z)$ of multiple variables. (E.g., temperature).

ii. Vector-valued function $\vec{F}(x, y, z)$ of multiple variables. (E.g., fluid flow, electric field.)

(b) One vector derivative operator:

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k} \, . \tag{15}$$

(c) Two ways to combine vectors: $\cdot$ and $\times$:

(d) The derivative of a scalar function $f(x, y, z)$ is the gradient vector: $\vec{\nabla} f$.

(e) Two derivatives for a vector field:

$$\vec{\nabla} \cdot \vec{F} \text{ and } \vec{\nabla} \times \vec{\mathrm{F}} \, . \tag{16}$$

The axioms[1] for this course were:

1. Mathematical potential is distributed equally among different groups, irrespective of geographic, demographic, and economic boundaries.

2. Everyone can have joyful, meaningful, and empowering mathematical experiences.

3. Mathematics is a powerful, malleable tool that can be shaped and used differently by various communities to serve their needs.

4. Every student deserves to be treated with dignity and respect.

The community agreement[2] for this course was:

> This course aims to offer a joyful, meaningful, and empowering experience to every participant; we will build that rich experience together by devoting our strongest available effort to the class. You will be challenged and supported. Please be prepared to take an active, critical, patient, creative, and generous role in your own learning and that of your classmates.

The goals for this course were:

1. Stay physically and mentally healthy and maintain intellectual and personal connection during a potentially difficult time.

2. I want to help you improve your problem solving skills, and mathematical confidence, and overall ability to use mathematics.

3. I want you to understand and know how to use partial derivatives, directional derivatives, double and triple integrals, and their applications.

4. I want you to gain experience using a computer to help you do mathematics.

5. I want you to understand and know how to use the main elements of vector calculus: the divergence, gradient, and curl; line and surface integrals; and Greens Theorem, Stokes Theorem, and the Divergence Theorem.

6. I want you to have fun while working hard and learning a lot.

[1] These axioms were written by Federico Ardila-Mantilla.
[2] Based on Ardila-Mantilla's community agreement for his classes.