Course Summary

Linear Algebra with applications to differential equations

College of the Atlantic. Winter 2019

Course Goals

1. I want you to learn what differential equations are and how to think about them: how they're used to model a range of phenomena and how to interpret their solutions.
2. I want you to gain a firm foundation in the basic concepts of elementary linear algebra, including: vectors, matrices, inverses, determinants, vector spaces, linear independence, subspaces, basic, dimension, rank, eigenvalues and eigenvectors.
3. I want you to gain an introduction to some of the basic analytic techniques used to analyze linear differential equations, including systems of linear equations.
4. I want you to gain mathematical confidence, appreciation, and "maturity". As part of this, I want you to continue to work toward developing a careful, systematic, and effective problem-solving style.
5. I want you to have fun and learn a lot.

Chronological List of Topics

1. Introduction to differential equations
2. Separable equations
3. Linear first-order equations; integrating factors
4. Introduction to linear systems
5. Matrices and row reduction
6. Matrix multiplication.
7. Matrix inverses
8. Determinants
9. Vector spacee and subspaces
10. Linear independence
11. Basis and dimension
12. Row and column spaces
13. Orthogonality and dot products
14. Eigenvectors and eigenvalues
15. Diagonalization and similarity
16. Second-order equations with constant coefficients
17. Systems of first-order differential equations
18. Relation between second-order equations and linear systems
19. Eigenvalue methods for solving linear systems of differential equations
20. Complex eigenvalues and eigenvectors
21. Deficient eigenvalues
22. Linearization and Jacobians

Facts about square matrices. The following statements about an $n \times n$ matrix A are equivalent:

1. A is nice[1].
2. A^{-1} exists.
3. $Ax = 0$ has only one solution: $x = 0$.
4. $Ax = b$ has a unique solution.
5. The determinant of A is non-zero.
6. The columns of A are linearly independent.
7. The rows of A are linearly independent.
8. The column space of A is $\mathbb{R}^n$.
9. The row space of A is $\mathbb{R}^n$.
10. None of the eigenvalues equal zero.
11. A is non-singular.
12. The reduced row echelon form is the $n \times n$ identity matrix.
13. The rank of A is n.
14. The echelon form of A has n pivots.

A few elementary matrix properties. Let A, and B be matrices. Assume all products are defined.

1. $(A^{-1})^{-1} = A$
2. $(A^{\mathrm{T}})^{\mathrm{T}} = A$
3. $(AB)^{-1} = B^{-1}A^{-1}$
4. $\det(AB) = \det(A)\det(B)$
5. $\det(A^{-1}) = 1/\det(A)$
6. It is not necessarily the case that $AB \neq BA$

[1] Not a standard mathematical term

Linear algebra lets us study systems of equations as if they were just one equation. Suppose x, a, and b are **numbers**. Then

1. 1 is the multiplicative identity. $1a = a$, for any a.
2. The inverse of a is $a^{-1} = \frac{1}{a}$. This means that $aa^{-1} = 1$.
3. The inverse exists unless $a = 0$.
4. Consider the equation $ax = 0$. The only solution to this equation is $x = 0$, unless $a = 0$, in which case there are infinitely many solutions.
5. If $ax = 0$ has a nonzero solution, then a is not invertible.
6. Consider the product ax. I can undo this and recover x by multiplying by the inverse: $a^{-1}ax = x$, unless $a = 0$.
7. The equation $ax = b$ has a solution for any b, unless $a = 0$, in which case it has no solutions.

Let x and b be vectors, and A be a **matrix**.

1. I is the identity matrix. $IA = A$, for any A.
2. The inverse of A is A^{-1}. This means that $AA^{-1} = I$.
3. The inverse of A exists unless $\det A = 0$.
4. Consider the equation $Ax = 0$. The only solution to this equation is $x = 0$, unless $\det A = 0$.
5. If $Ax = 0$ has a nonzero solution, then A is not invertible.
6. Consider the product Ax. I can undo this and recover x by multiplying by the inverse: $A^{-1}Ax = x$, unless $\det A = 0$.
7. The equation $Ax = b$ has a solution for any b, unless $\det A = 0$.