Activity 2.3.3 In this activity, we will look at the span of sets of vectors in $\mathbb{R}^3$.

a. Suppose $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. Give a geometric description of Span$\{\mathbf{v}\}$ and a rough sketch of $\mathbf{v}$ and its span in Figure 2.3.10.

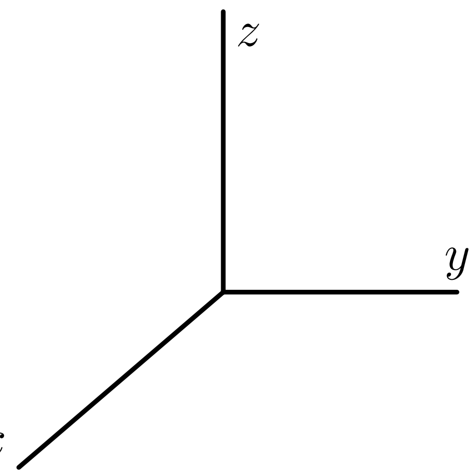

Figure 2.3.10 A three-dimensional coordinate system for sketching $\mathbf{v}$ and its span.

b. Now consider the two vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Sketch the vectors below. Then give a geometric description of Span$\{\mathbf{e}_1, \mathbf{e}_2\}$ and a rough sketch of the span in Figure 2.3.11.

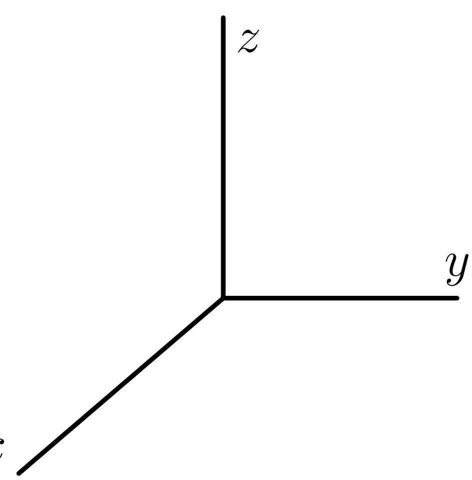

Figure 2.3.11 A coordinate system for sketching $\mathbf{e}_1$, $\mathbf{e}_2$, and Span$\{\mathbf{e}_1, \mathbf{e}_2\}$.

c. Let's now look at this situation algebraically by writing write $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Determine the conditions on b_1, b_2, and b_3 so that $\mathbf{b}$ is in Span$\{\mathbf{e}_1, \mathbf{e}_2\}$ by considering the linear system

$$\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix} \mathbf{x} = \mathbf{b}$$

or

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Explain how this relates to your sketch of Span$\{\mathbf{e}_1, \mathbf{e}_2\}$.

d. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}.$$

1. Is the vector $\mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$ in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$?

2. Is the vector $\mathbf{b} = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$ in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$?

3. Give a geometric description of Span$\{\mathbf{v}_1, \mathbf{v}_2\}$.

e. Consider the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}.$$

Form the matrix $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$ and find its reduced row echelon form.

What does this tell you about Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

f. If the span of a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is $\mathbb{R}^3$, what can you say about the pivot positions of the matrix $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n \end{bmatrix}$?

g. What is the smallest number of vectors such that Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \mathbb{R}^3$?