Lab Two: Introduction to Markov Models

Linear Algebra

College of the Atlantic

Due Friday, April 26, 2024

Instructions: Complete the following exercises in groups of two students; submit a single report for your group. If you want, you can write directly on the handout. Be sure to include complete explanations of the work you have done and justification for your conclusions. Scan your work and submit on google classroom. Please make sure that both lab partners' names are on the assignment.

Use `https://cocalc.com` for sage as needed.

These problems are based on exercises written by Matt Boelkins, which are based on problems from Understanding Linear Algebra, by David Austin.

Introduction. In what follows, we investigate how linear algebra can be used to model a probabilistic scenario that describes the distribution of bicycles among possible locations in a city. Initially, we'll work with a simple situation where there are just two locations where bicycles may be rented and returned.

Suppose that Mount Desert Island decides to start a bike-sharing program and opens two locations where bicycles may be borrowed, say B and C. Over the first few months, the city tracks the number of bicycles in each location at the end of the day, and over time finds that 70% of bikes rented at location B at the start of the day are returned to location B at the end of the day (and all the remaining bikes rented at B are returned to C), while 40% of bikes rented at location C are returned to location C (and all the remaining bikes rented at C are returned to B). The managers of the bike rental program become interested in the question: *what is an ideal distribution of bikes in the two locations*?

To study this situation, we introduce the vector $\vec{x}_k = \begin{bmatrix} B_k \\ C_k \end{bmatrix}$ to represent the number of bicycles in each location at the start of day k. In particular, the value of B_k is the number of bicycles at location B at the start of day k, and C_k is the number of bikes at C at the start of day k. The information above about which percentage of bikes are returned to which location helps us see how linear equations are involved in finding the distribution of bikes on different days in the future. For example, since 70% of the bikes rented from location B are returned to location B, and 60% of bikes rented from location C are returned to location B, it follows that the number of bikes at location B on day $k+1$ must be

$$B_{k+1} = 0.7B_k + 0.6C_k \tag{1}$$

Recall that 70% of bikes rented at location B at the start of the day are returned to location B at the end of the day (with the rest being returned to C), while 40% of bikes rented at location C are returned to location C (with the rest being returned to B).

1. First, we explore some numerical examples.

   (a) If there are initially 100 bikes at location B and no bikes at location C, how many bikes are at each location at the end of one day? Show your computations and thinking.

   (b) If instead there are initially 100 bikes at location C and none at location B, how many bikes are at each location at the end of one day? Show your computations and thinking.

   (c) What if there are initially 100 bikes at B and 100 at C? How many bikes are at each location at the end of the day? Again, show your computations and thinking.

2. Next, let's use linear algebra ideas and notation to make the computations easier.

   (a) (1) Find an equation for C_{k+1} in terms of B_k and C_k that is similar to Equation (1) at the end of the introduction on the previous page. State your equation below.

   (b) Now, write both the equations for B_{k+1} and C_{k+1} in the space below by filling in the blanks:

$$B_{k+1} = \_\_\_ B_k + \_\_\_ C_k$$
$$C_{k+1} = \_\_\_ B_k + \_\_\_ C_k$$

   For what matrix A is it true that $\vec{x}_{k+1} = A\vec{x}_k$? That is, for what matrix A is it true that $\begin{bmatrix} B_{k+1} \\ C_{k+1} \end{bmatrix} = A \begin{bmatrix} B_k \\ C_k \end{bmatrix}$? State the matrix A in the space below, and then test your matrix by computing $A\begin{bmatrix} 100 \\ 0 \end{bmatrix}$, $A\begin{bmatrix} 0 \\ 100 \end{bmatrix}$, and $A\begin{bmatrix} 100 \\ 100 \end{bmatrix}$, and comparing your results in #1 above.

Your work above now enables you to do computations such as "$\vec{x}_2 = A\vec{x}_1$", so that if you know the distribution $\vec{x}_1$ on Day 1, you can thus find the distribution $\vec{x}_2$ on Day 2.

If you are at all uncertain about whether or not you have the correct matrix A, please check your work with me (Dave) before proceeding with the following questions. You should use Sage appropriately for any computational work that follows.

3. Having established that $\vec{x}_{k+1} = A\vec{x}_k$, we now use the matrix A to find $\vec{x}_k$ for other values of k given information a certain $\vec{x}_k$.

   (a) Suppose that on a Monday morning (say $\vec{x}_1$), there are 800 bicycles at location B and 800 bicycles at location C. How many bikes are there at the respective locations on the following Tuesday morning? On Wednesday morning? On Thursday morning? Show a summary of the matrix computations and vectors that enabled you to make your conclusions.

   (b) Suppose that on a Saturday morning (say $\vec{x}_6$), there are 998 bikes at location B and 502 bikes at location C. How many bikes were at each location on the preceding Friday morning? On Thursday morning? Show a summary of your computations and write at least one sentence to explain your thinking.

4. Next we investigate how some special vectors related to the matrix A (the same matrix you've been using since #2) enable us to study the long-term behavior of the distribution of the bicycles at locations B and C. Let $\vec{v}_1$ and $\vec{v}_2$ be given by

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

   (a) Compute $A\vec{v}_1$. How does this vector compare to $\vec{v}_1$?

   (b) Compute $A\vec{v}_2$ and $0.1\vec{v}_2$. What do you notice?

   (c) Suppose that for some initial distribution $\vec{x}_1$, we can write $\vec{x}_1$ as the linear combination

$$\vec{x}_1 = c_1\vec{v}_1 + c_2\vec{v}_2$$

   Use the Linearity Principle (Proposition 2.2.3) and your work in (a) and (b) above to explain why

$$A\vec{x}_1 = c_1\vec{v}_1 + 0.1c_2\vec{v}_2$$

   (*Note that your most recent result above enables you to write $\vec{x}_2$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$.*)

   (d) Next, explain why

$$\vec{x}_3 = c_1\vec{v}_1 + 0.1^2c_2\vec{v}_2.$$

   What can you say about $\vec{x}_4$? About $\vec{x}_5$?

   (e) Now suppose that $\vec{x}_1$ is given by a distribution of 800 bikes at location B and 700 bikes at location C. Find scalars c_1 and c_2 so that $\vec{x}_1 = c_1\vec{v}_1 + c_2\vec{v}_2$, where $\vec{v}_1$ and $\vec{v}_2$ are the same two vectors we've been working with since the start of #4.

(f) Use your results from #4c and #4d to determine the distribution of bikes at locations $\boldsymbol{B}$ and $\boldsymbol{C}$ on days 2, 3, 4, and 5 by finding $\vec{x}_2$, $\vec{x}_3$, $\vec{x}_4$, and $\vec{x}_5$.

(g) After a long time, how will the 1500 bikes originally distributed with 800 at $\boldsymbol{B}$ and 700 at $\boldsymbol{C}$ be distributed? Why?

(h) If the city wants to build covered locations at $\boldsymbol{B}$ and $\boldsymbol{C}$ to house the bikes they expect to be at each location, what would be an ideal number of bikes for each location to hold, assuming the program involves 1500 total bikes?