Lab Four: The Fibonacci Sequence

Linear Algebra

College of the Atlantic

Due Friday, June 7, 2024

Instructions: Complete the following exercises in groups of two students; submit a single report for your group. If you want, you can write directly on the handout. Be sure to include complete explanations of the work you have done and justification for your conclusions. Scan your work and submit on google classroom. Please make sure that both lab partners' names are on the assignment.

This exercise is designed to be done without the use of sage.

Introduction. Remember the Fibonacci sequence? It's kinda fun, right? In this lab we'll derive some interesting and non-obvious results about the Fibonacci sequence.

The Fibonacci sequence $F_0, F_1, F_2, \ldots$ is defined by the following recursive formula:

$$F_{k+1} = F_k + F_{k-1} \ , \tag{1}$$

with

$$F_0 = 0 \ , \text{ and } F_1 = 1. \tag{2}$$

1. Using Eqs. (1) and (2), write down the first dozen Fibonacci numbers.

2. How long do you think it would take you to figure out F_{1000}? Quite a while, right?

3. It would be neat to find a direct formula for F_n, wouldn't it? That way you could calculate F_{1000} without having to do addition 1000 times.

4. We'll use linear algebra to find a direct formula for F_k. We begin by defining a vector

$$\vec{u}_k = \begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix} \ . \tag{3}$$

That is, $\vec{u}_k$ is a vector that "bundles together" two consecutive Fibonacci numbers: F_k and F_{k-1}. Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \ . \tag{4}$$

(a) What happens if you multiply A by $\vec{u}_k$? Show that

$$A \begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix} = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} . \tag{5}$$

(b) Now let's do this with numbers. Using Eq. (2), we know that

$$\vec{u}_1 = \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} . \tag{6}$$

Calculate F_2 by evaluating $A\vec{u}_1$.

(c) Calculate F_3 by evaluating $A\vec{u}_2$.

(d) Calculate F_4 by evaluating $A\vec{u}_3$.

(e) Generalizing, write a formula for $\vec{u}_{n+1}$ in terms of A and $\vec{u}_1$.

5. The formula you wrote down above should involve a power of the matrix A. We know how to find a simple formula for the power of a matrix—diagonalization. As a first step, find the eigenvalues of A:

   (a) Write down the characteristic polynomial for A.

   (b) Now solve for the eigenvalues [1] by hand. Do not approximate any square roots.

[1] $\lambda_1 = (1 + \sqrt{5})/2$ and $\lambda_2 = (1 - \sqrt{5})/2$.

6. Now we need to find the eigenvectors. This will take a bit of algebra. It may be helpful to define $\phi = \lambda_1$. Then it follows that $1 - \phi = \lambda_2$. Also, note that ϕ isn't just any random number: we know that ϕ solves the characteristic equation. This will prove useful when simplifying some algebra.

Check with me if you're stuck or just want to make sure you've got the right answer.

7. Write down matrices[2] P and D such that

$$A = PDP^{-1}\,. \tag{8}$$

8. Use this result to come up with an expression for

$$\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}\,. \tag{9}$$

9. You now have an expression for F_n, the n^{th} Fibonacci number. Simply your expression. You should end up with:

$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}\,. \tag{10}$$

10. by hand, use this formula to evaluate F_2.

11. Use a calculator or computer and Eq. (10) to evaluate F_{1000}.

[2]btw, if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\,,\ \text{then } A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\,. \tag{7}$$