Theory and Applications of Complex Networks

Class Five

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1. Introduction
2. Power-law Degree Distributions
3. Network Growth Models

Networks Growth and Dynamics

- “Things are the way they are because they got that way.” (Richard Levins)
- The Watts-Strogatz model sheds light on a static network.
- The WS network is *not* intended to be a model of how networks actually grow.
- The WS does, however, capture some aspects of some already-formed, real-world networks.
- Today, we start thinking about how networks might form.
- What models are there for network growth, and what do they tell us?

Empirical Observation: Power-Law Degree Distribution

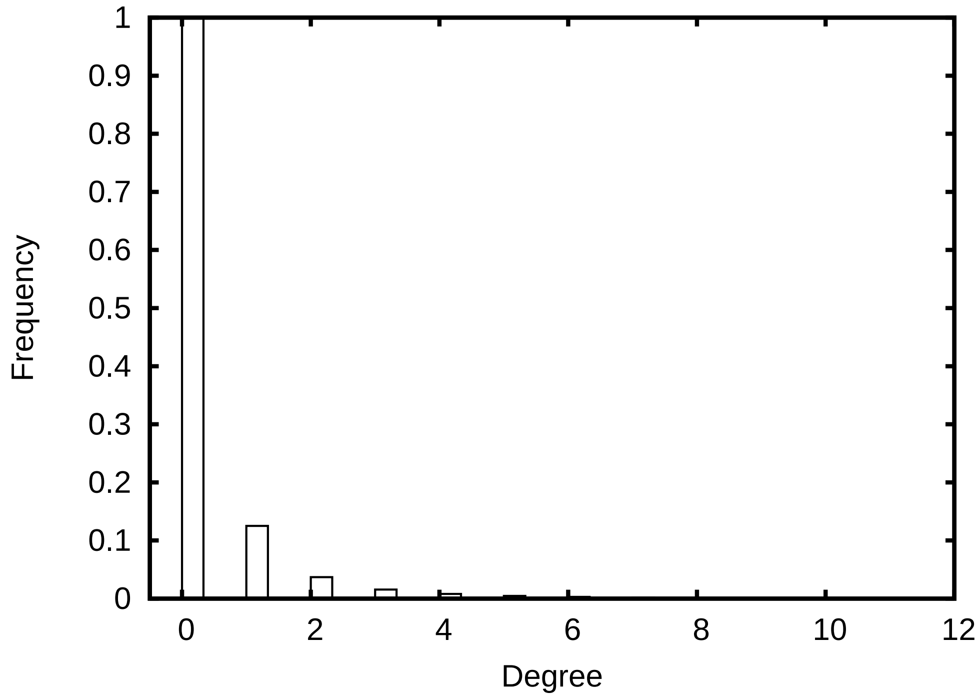
- It is well fairly established that many networks have a power-law degree distribution.
- A power-law distribution is one of the following form:

$$P(k) = ck^{-\alpha} \;, \tag{1}$$

where α is a positive constant, usually between 1 and 3.

- c is a constant that is adjusted to normalize the distribution.
- Often a distribution is called a power-law distribution even if it doesn't have exactly the above form, so long as it has this form for large k.
- What's usually of interest in these types of distribution is their large-k behavior.

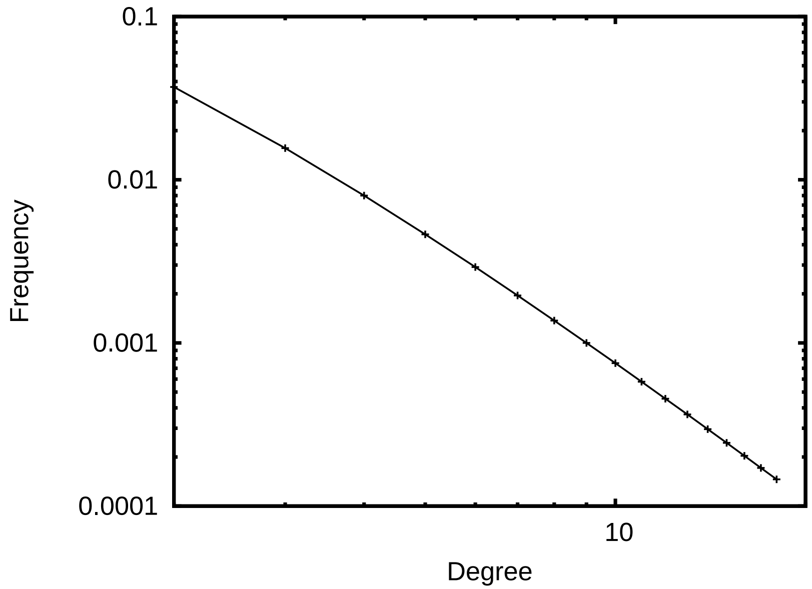
What do Power Law Distributions Look Like?



- Above is an (unnormalized) power law distribution for $\alpha = 3$.
- Note the relatively rapid decay of the degree k.
- However, the decay is much, much, much slower Poisson. For large k,

$$k^{-\alpha} >> \frac{1}{k!} \,. \tag{2}$$

What do Power Law Distributions Look Like?



- A power-law distribution is linear on a log-log plot.
- To see this, take the log of both sides of Eq. (2):

$$\log p = \log c - \alpha \log k \ . \tag{3}$$

- The slope of the line is the exponent α.
- Note: This is resoundingly not the best way to estimate α. More on this later.

Some Data

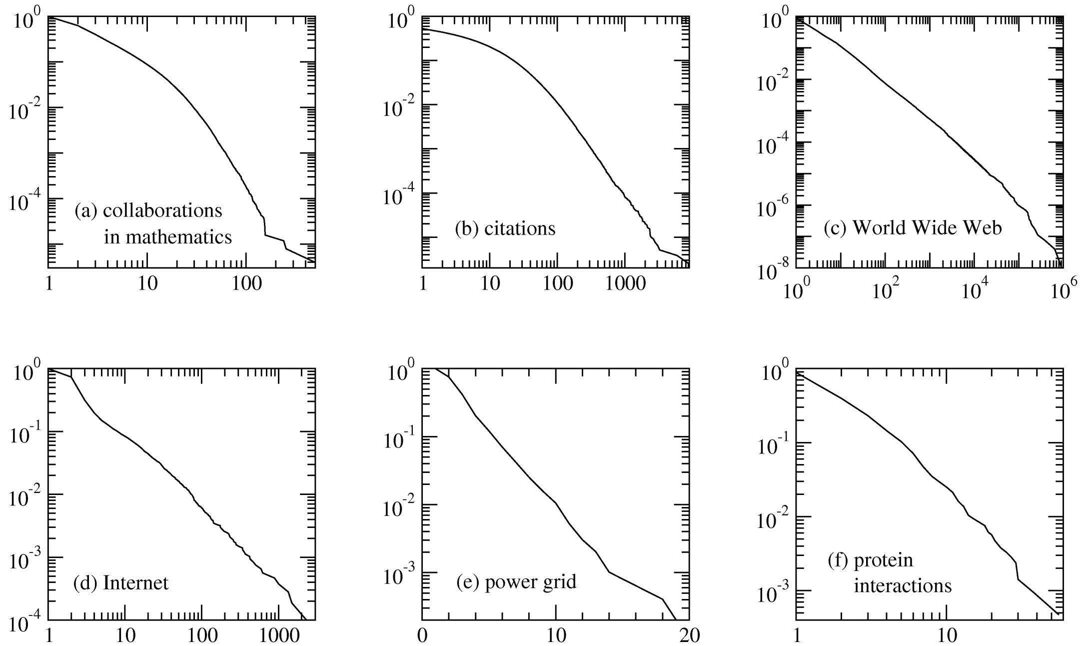


FIG. 6 Cumulative degree distributions for six different networks. The horizontal axis for each panel is vertex degree k (or in-degree for the citation and Web networks, which are directed) and the vertical axis is the cumulative probability distribution of degrees, i.e., the fraction of vertices that have degree greater than or equal to k. The networks shown are: (a) the collaboration network of mathematicians [182]; (b) citations between 1981 and 1997 to all papers cataloged by the Institute for Scientific Information [351]; (c) a 300 million vertex subset of the World Wide Web, *circa* 1999 [74]; (d) the Internet at the level of autonomous systems, April 1999 [86]; (e) the power grid of the western United States [416]; (f) the interaction network of proteins in the metabolism of the yeast *S. Cerevisiae* [212]. Of these networks, three of them, (c), (d) and (f), appear to have power-law degree distributions, as indicated by their approximately straight-line forms on the doubly logarithmic scales, and one (b) has a power-law tail but deviates markedly from power-law behavior for small degree. Network (e) has an exponential degree distribution (note the log-linear scales used in this panel) and network (a) appears to have a truncated power-law degree distribution of some type, or possibly two separate power-law regimes with different exponents.

Figure Source: Newman, SIAM Reviews, 45(2): 167-256, 2003.

Why Might we Care About Power Laws?

- Much more about this later. For now, just a few initial thoughts.
- Power laws are very different from Poisson and Gaussian distributions.
- The probability of extreme events is much, much larger for PLs than Gaussians and Poissons.
- Power laws are "scale-free" or fractal
- This suggests that a common mechanism may be responsible for the behavior across all scales.
- I.e., there is a single explanation that explains both poorly and highly connected nodes.
- More generally, some think that power laws are "deep" and indicate some special type of organization or simplicity.
- Power laws—"the long tail"—have become a powerful metaphor or stylized fact.

How might Power Laws Form?

- Is there a model of network growth that exhibits a power law degree distribution?
- Yes. In 1999, Barabási and Albert (re)introduce a model for growth that produces power laws.
- Barabási and Albert, ”Emergence of scaling in random networks”, Science, 286:509-512, October 15, 1999.
- There is quite a long pre-history to this model. It turns out that their basic idea goes back to at least 1925, and their model is a special case of other models that had been previously published.
- There has also been much follow-up work and some significant critique of this model
- More on pre- and post-history later.

Rich-get-richer

- There is a class of growth models—not just for networks—based on the following idea.
- Nodes with more links are more likely to get more links.
- This idea goes by many different names:
  - Cumulative advantage (Simon)
  - Rich get richer
  - Preferential attachment (Barabási and Albert)
  - Matthew effect (Merton)
  - Yule process
- Matthew 25:29. “For unto every one that hath shall be given, and he shall have abundance: but from him that hath not shall be taken away even that which he hath.”

The “Barábasi-Albert” Model

- As noted above, there are lots of variations and precursors to this model.
- Here is the simplest version of the model:
  1. Nodes are added to the network one at a time. Each node makes m links to existing nodes.
  2. The nodes are randomly connected to the existing nodes, with a probability that is proportional to the number of links that node has.
- We will simulate this model using yarn.
- I’ll then “hand-wavingly” derive a power law degree distribution on the blackboard.