Differential Equations

Homework Four

Due 23 May 2008

1. Consider linear systems of the following form:

$$\frac{dx}{dt} = ax + by \quad (1)$$
$$\frac{dy}{dt} = cx + dy \ . \quad (2)$$

Or, using matrix notation:

$$\vec{X}' = A\vec{X} \ , \quad (3)$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ , \quad (4)$$

and

$$\vec{X} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \ . \quad (5)$$

For three systems below, do the following:

- Find the solution of the IVP. To do so, find the eigenvalues and eigenvectors.
- Sketch $x(t)$, $y(t)$, and the solution on the x, y phase plane.
- Classify the equilibrium point.

(a)

$$A_1 = \begin{pmatrix} 3 & 2 \\ 0 & -2 \end{pmatrix} \ , \quad (6)$$

$$\vec{X}_0 = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \ . \quad (7)$$

(b)

$$A_2 = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} \ , \quad (8)$$

$$\vec{X}_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \ . \quad (9)$$

(c)

$$A_3 = \begin{pmatrix} -1 & 2 \\ -1 & -1 \end{pmatrix}, \tag{10}$$

$$\vec{X}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{11}$$

2. Chapter 7, problem 7.

3. Optional: Consier a 2×2 matrix that is symmetric. Show that its eigenvalues must be real.

4. Optional (recommended for linear algebra students): Suppose the matrix A has the complex eigenvalue $\lambda = \alpha + i\beta$, where $\beta \neq 0$. Let $\vec{u}$ be an eigenvector. This vector will have complex entries. Write this eigenvector as $\vec{u} = \vec{u_1} + i\vec{u_2}$, where $\vec{u_1}$ and $\vec{u_2}$ are real. Show that $\vec{u_1}$ and $\vec{u_2}$ are linearly independent. (From Blanchard, Devaney, and Hall, *Differential Equations*, second edition, Brooks-Cole, 2002. p. 299.)