Differential Equations
Homework Three
Due Friday, October 31, 2014

1. Consider a linear system of two ODEs:

$$\frac{d\vec{Y}}{dt} = A\vec{Y} \,, \tag{1}$$

where A is the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \,, \tag{2}$$

and

$$\vec{Y} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \,. \tag{3}$$

   (a) Find the eigenvalues and eigenvectors of A.[1]

   (b) Write down two linearly independent solutions to Eq. (1).

   (c) Show that

$$\vec{Y_3}(t) = e^{t} \begin{pmatrix} 2 \\ -2 \end{pmatrix} \tag{4}$$

   is a solution to Eq. (1).

   (d) Show that

$$\vec{Y_3}(t) = e^{4t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \tag{5}$$

   is not a solution to Eq. (1).

   (e) Show that

$$\vec{Y_3}(t) = e^{5t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} \tag{6}$$

   is not a solution to Eq. (1).[2]

   (f) Determine the solution to Eq. (1) that has $\vec{Y}(0) = (0, 1)$.

   (g) How would you classify the equilibrium point at the origin?

   (h) Using wolframalpha to assist you, sketch the phase plane for this system. Include the two straight-line solutions that you found in question 1b.

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[1] You should find that $\lambda_1 = 5$ and $\lambda_2 = 1$. The eigenvectors are

$$\vec{V_1} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \,, \text{ and } \vec{V_2} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \,.$$

[2] The point of the last few exercises is to show that both the eigenvalue and the eigenvector have to be right in order to $Y(\vec{t})$ to be a solution.

2. Consider again the Lotka–Volterra system:

$$\frac{dx}{dt} = Ay - Bxy \ , \tag{7}$$

$$\frac{dy}{dt} = Cxy - Dy \ . \tag{8}$$

   (a) Determine the two equilibria for this system.
   (b) Find the Jacobian matrix.
   (c) Evaluate the Jacobian matrix at the non-zero equilibrium point and determine its eigenvalues. What do the eigenvalues tell us about behavior near the equilibrium point. Is this consistent with what we have seen numerically?
   (d) Evaluate the Jacobian matrix at the zero equilibrium point. Choose $A = 4, B = 3, C = 2, D = 4$, and find the matrix's eigenvalues. What do the eigenvalues tell us about behavior near the equilibrium point? Does this make sense biologically?

3. Optional, but recommended. In the previous assignment we modified the prey term in the Lotka–Volterra equations so that they grow logistically in the absence of predators:

$$\frac{dx}{dt} = Ax\left(1 - \frac{x}{N}\right) - Bxy \ . \tag{9}$$

   In the following, let $A = 2$, $B = 0.5$, $C = 0.2$, $D = 1$, and $N = 20$.

   (a) Solve for the non-zero equilibrium point.
   (b) Find the Jacobian matrix.
   (c) Evaluate the Jacobian matrix at the non-zero equilibrium.
   (d) Find the matrix's eigenvalues. What do the eigenvalues tell us about the behavior of the model near the equilibrium? Is this consistent with what you saw numerically?

4. **Optional, but recommended.** Here is a clever way to prove/verify Euler's formula. Define the following function:

$$g(x) = (\cos x + i \sin x)\, e^{-ix} \ . \tag{10}$$

   (a) Show that $\frac{dg}{dx} = 0$. This means that $g(x)$ is a constant function. Thus, it is the same for all values of x.
   (b) Show that $g(0) = 1$.
   (c) Since $g(x)$ is constant, it must be that $g(x) = g(0)$. Use this fact to derive Euler's formula.

5. **Optional:** Use complex exponentials to derive "double angle formulas." That is, determine expressions for $\cos(2x)$ and $\sin(2x)$ in terms of $\cos(x)$ and $\sin(x)$. To do so, use the fact that:

$$e^{2ix} = e^{ix}e^{ix} \ . \tag{11}$$

   Use Euler's formula on each complex exponential. Multiply, simplify, and group the real parts and the imaginary parts. Like magic, you will have derived two trig identities. Many other trig identies can be derived via similar hijinks.

6. **Optional:** Here is another handy use of Euler's formula,

$$e^{ix} \; = \; \cos(x) + i \sin(x) \; . \tag{12}$$

Consider the following integral:

$$\int e^{ax} \sin(bx) \, dx \; . \tag{13}$$

Ordinarily, you would do this integral using integration by parts. But there is another way to do it. Re-write the $\sin(bx)$ term in the integrand using Euler's formula. I.e.,

$$\sin(bx) \; = \; \Im e^{ibx} \; , \tag{14}$$

where $\Im$ means "imaginary part of." You have now converted the integral into something involving only exponentials. Do the integral and you will get an algebraic expression. Solve for the imaginary part of this expression, and you'll have the answer to the integral of Eq. (13). To do so, you'll need to use Euler's formula in reverse, and will also need to get rid of any i's in the denominator of any fractions. This method for evaluating an intergral is a big algebra intensive, but it does avoid having to integrate by parts.