Differential Equations
College of the Atlantic
Homework One
Due Friday, January 17, 2020

Please submit your code via google classroom. For your plots, either give me hard copies or submit through google classroom. If at all possible, please don't email me any homework. Thanks!

In this assignment you will carry out an analysis of the logistic differential equation:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right) . \tag{1}$$

This equation describes how a population $P(t)$ varies over time. In the equation k and N are positive parameters. You will analyze this in several different but complementary ways, much we have done with Newton's law of cooling in class. We will restrict our analysis to non-negative values of P.

1. Sketch the right-hand side of Eq. (1). Use $k = 2$ and $N = 500$.

2. Make qualitatively accurate sketches of $P(t)$ for initial populations of 100, 400, and 700.

3. What is the biological significance of the values of k and N in the equation? (I'm just looking for one or two sentences, not a lengthy exegesis.)

4. Verify that the following is a solution to Eq. (1):

$$P(t) = \frac{NP_0}{P_0 + (N - P_0)e^{-kt}} , \tag{2}$$

   where P_0 is the initial population. To do so, you'll need to plug the above expression into Eq. (1) and show that the equation is true. It'll involve a bit of differentiation and algebra.[1][2]

5. Use python to plot Eq. (2) for the three values of P_0 that you used in your sketches for Question 2. Make your plot nice. Include a key, axis labels, etc.

6. Write a program that implements Euler's method for solving ordinary differential equations. Your program should be clearly written and be well commented.

[1] This will probably be the most actual calculus we'll do all term.

[2] Think of this problem as almost optional. If you want a potentially challenging (but not devious) derivative practice/review, this problem is for you. But don't get hung up or stuck on it. This problem is not that important; we won't really do anything like this again in the course. The other problems are much more central.

7. Use your Euler program to make plots of your Euler solution for $\Delta t = 1$ and $\Delta t = 0.1$ from $t = 0$ to 10. Make plots of these two solutions together with the exact solution, Eq. (2). Use $P_0 = 100$.

8. Optional: Try Euler with $\Delta t = 2$ and plot the solution.

9. Use your Euler program to experiment with different values of Δt. How small a Δt is small enough? Briefly explain.

10. Use your Euler program to plot solutions to Eq. (1) for the three P_0 values you used in Question 2. Plot all three solutions on the same axes. (Note that you've now made this same plot three different ways: a qualitative sketch by hand, an exact plot using python and a formula, and an essentially exact plot using python and Euler's method.)

Optional, recommended for people who want to do a little bit of calculus. Consider Newton's Law of Cooling:

$$\frac{dT}{dt} = -k(T - A) \; . \qquad (3)$$

1. Define a new function $y = T - A$, the difference between the temperature T of the object and the ambient temperature A. After plugging in you should get another differential equation where $y(t)$ is the unknown function instead of $T(t)$.

2. Hey! That new differential equation looks familiar. Write down its general solution.

3. Then use the $y(t)$ you just figured out to write down the solution $T(t)$.