Differential Equations
Homework Two
Due Friday, January 31, 2020

1. The re-scaled SIR model is

$$\frac{dX}{d\tau} = -R_0 XY \,, \tag{1}$$

$$\frac{dY}{d\tau} = R_0 XY - Y \,, \tag{2}$$

$$\frac{dZ}{d\tau} = Y \,, \tag{3}$$

Where X, Y, Z are the fraction of the population that is susceptible, infected, and removed, respectively. The quantity R_0 is the basic reproductive rate, and τ is the rescaled time: $\tau = \gamma t$, where the mean duration of infection is $1/\gamma$. This means that if τ increases by 1, the time t will have increased by one mean infection time.

We will consider of Ebola,[1] for which the mean infection time is around 4 days. So $\gamma = 1/4$. Estimates of the basic reproduction rate vary; we'll use $R_0 = 1.8$.[2]

(a) Suppose you are studying a community of 1000 people and initially 10 people are infected. Make a plot of $X(t)$, $Y(t)$, and $Z(t)$. How many people will get sick? At what time (in days) is the outbreak the worst—i.e., when are the largest number of people sick?

(b) An outbreak of Ebola occurs in a community of 2000, starting with a single infected person. Over the course of the epidemic, a total of 1500 people get sick. Use this information to estimate R_0 for Ebola in this community.

2. In this series of exercises we'll extend the basic SIR model in a way that might make it more applicable to measles. (**Note: Figuring out time units for the various parameters will require some thought and care.**) Let's use an R_0 of 15 and assume a mean infection time of 6 days.[3]

(a) Add a birth rate term to the basic SIR model. Choose a semi-realistic value, thinking carefully about units. Use α for the birthrate. We'll assume that the birth rate is constant, independent of the total population size. Let's assume that the each person in the model has a 10% change of acquiring a sibling each year.

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[1] Ebola is usually modeled with equations that are a bit more complex than the basic SIR model considered here.

[2] See, e.g., Fisman D, Khoo E, Tuite A. Early Epidemic Dynamics of the West African 2014 Ebola Outbreak: Estimates Derived with a Simple Two-Parameter Model. *PLOS Currents Outbreaks.* 2014 Sept. 8. `http:goo.gl/m20MGE`.

[3] The value of R_0 varies widely from country to country. Guerra, Fiona M., et al. "The basic reproduction number (R0) of measles: a systematic review." *The Lancet Infectious Diseases* (2017). I'm not sure if 6 is a realistic mean infection time, but it's probably close.

(b) Outbreaks of measles are periodic—or at least they were before vaccines. One possible explanation for this is that the contact rate b changes. When school is in session, b is large. When school is out of session, b is small. Incorporate this into your model. To do so, let's assume that the contact rate is such that $R_0 = 5$ for six months of the year and has a value of $R_0 == 15$ for the other six months of the year. Show the result of your code, using a total population of $10,000$ and with 10 initially infected people. Let the simulation run for several years. Explain what you observe.

3. **Optional.** The SIRS model is:

$$\frac{dS}{dt} = -\beta SI + \nu R \,, \tag{4}$$

$$\frac{dI}{dt} = \beta SI - \gamma I \,, \tag{5}$$

$$\frac{dR}{dt} = \gamma I - \nu R \,. \tag{6}$$

Let N denote the total population: $N = S + I + R$. Solve for the non-zero equilibrium for this system. To do so, set the above three equations equal to zero and solve for S, I, and R. Use your code to test to see if your answers are correct. Choose parameter values (β, γ, and ν) and N, run your code, and see what the long-term steady state is. The S, I, R values should agree with the equilibrium values that you calculated by hand.