Spacetime

Physics II: Modern Physics

Tuesday and Wednesday, April 5 and 6, 2022

College of the Atlantic

Do this worksheet in parallel with the space worksheet. Work in pairs if you wish. There is nothing to hand in.

1. What *is* the spacetime interval? Your answer should be a sentence, not a formula.

2. What is a formula for the spacetime interval Δs between two points in a particular reference frame? We will view this formula as fundamental. We'll use it as a starting point to for spacetime geometry.

3. In a particular reference frame, let point A be at the origin and let point B have the coordinates $x = 3, t = 5$.

   (a) What is the spacetime interval between A and B?

   (b) In another reference frame suppose that $x = 0$. In this coordinate system, what is t?

   (c) In another reference frame suppose that $x = -6$. In this reference frame, what is t?

   (d) In another reference fraime, suppose that $x = 5$. In this reference frame, what is t?

4. Illustrate this state of affairs on a spacetime diagram.

   (a) Draw a spacetime diagram on which B has the coordinates $x = 3, t = 5$.

   (b) On these axes, draw all the other t, x values that you found in the previous problem.

   (c) The set of all possible t, x values for B makes what shape? Why? Answer both mathematically and physically.

5. In another coordinate system the point B has a value of $t = 3$. In this coordinate system, what is x?

6. Out of all the possible other coordinate systems, in what coordinate system is t the smallest? Physically or geometrically, how would you describe this coordinate system?

7. If we know s and either t or x, we can use the distance equation to figure out whichever of t and x we don't know. You've done this repeatedly in the problems above. But sometimes we might know the distance d and an "angle" α. (The word angle is in quotes because α is a sort of generalized angle and is not the same as the angles that you usually think about.) This scenario is illustrated below.

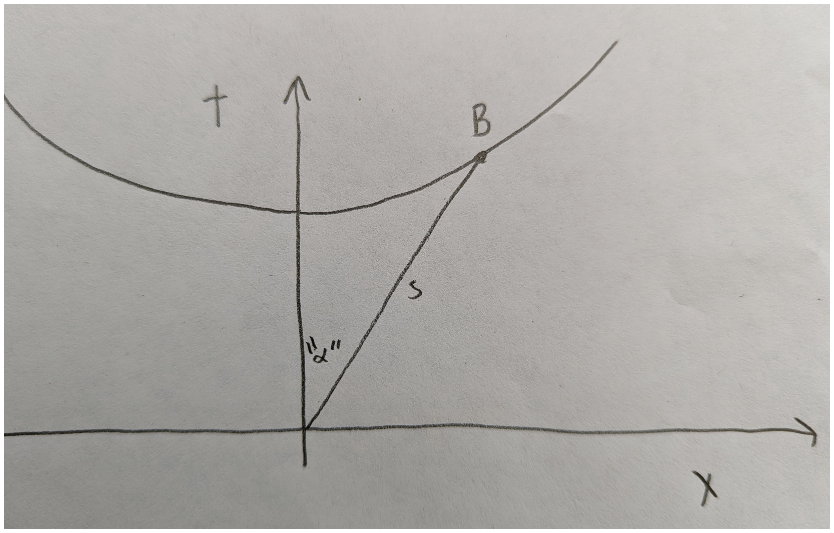

Figure 1: A hyperbola. What are the spacetime coordinates x and t of point B?

(a) Write down formulas for t and x in terms of α and Δs. Hmmm ... the answer doesn't involve sine and cosine. But let's invent some new functions—hyperbolic trig functions!—called $\cosh(\alpha)$ and $\sinh(\alpha)$. Use these newly invented functions to write down formulas for x and y.

(b) What *are* sinh and cosh? How are they defined? Hmm..... Well, let's just take your answers to the previous problems as the definition of sine and cosine. It would be nice to have a formula for sinh and cosh. But your calculator (probably) has buttons for these functions, so that's almost as good as a formula. Use your calculator to:

   i. Find x and t if $\alpha = 1$ degrees and $\Delta s = 4$.

   ii. Find α if $x = 3$ and $t = 5$.

(c) Use your answers to Question 7a and the distance formula to show that:

$$\cosh^2(\theta) - \sinh^2(\theta) = 1 \ . \tag{1}$$

(d) What *is* α, anyway? By analogy with the correct way to define "normal" angles θ, come up with a correct way to define α.

(e) In this physics setting, the "angle" has a physical interpretation. Let v be the speed of an object that goes from event A (the origin) to event B at constant velocity. Confince yourself that $\alpha = \tanh^{-1}(v)$.

(f) Btw, it turns out that there are (relatively) simple formulas for the hyperbolic trig functions:

$$\sinh(x) = \frac{e^x - e^{-x}}{2} , \tag{2}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} , \tag{3}$$

Use these formulas to:

i. Determine values for $\sinh(0)$ and $\cosh(0)$.

ii. Show that

$$\cosh^2(x) - \sinh^2(x) = 1 . \tag{4}$$