Chapter 19

Complex Numbers

In order to work our way to the Mandelbrot set, we'll need to put Julia sets aside for a moment and focus on complex numbers. In the subsequent chapter we will consider dynamical systems where the numbers we iterate are complex. Thus far, we've been iterating only real numbers. Iterating simple functions but with complex numbers will then lead us quickly to the Julia and Mandlebrot sets.

19.1   The Square Root of Negative One

What is the square root of negative one? That is, is there any x such that

$$x^2 = -1 ? \tag{19.1}$$

The answer to this equation can't be $x = 1$, since $1^2 = 1$. And the answer also isn't -1, since $(-1)^2 = (-1)(-1) = 1$. So we're stuck. The way to get out of this bind is to simply define a new number. We'll call this new number i, and it is defined by the fact that i squared is negative one:

$$i^2 = -1 . \tag{19.2}$$

Equivalently, this says that i is the square root of negative one:

$$i = \sqrt{-1} . \tag{19.3}$$

The number i is often referred to as an imaginary number. However, the term **complex number** is more commonly used, at least in mathematics

circles, for numbers of this sort. **Real numbers** are those numbers that are not complex. Thus far we've been working exclusively with real numbers. The number line, stretching from minus infinity to positive infinity, contains all the real numbers. Integers are a subset of the real numbers.

I think that the word "imaginary" carries some metaphysical or cosmic baggage that isn't helpful. After all, all numbers are imaginary—they're constructs and idealizations of ideas in people's heads. Numbers aren't tangible things that you can bump into in the physical world like a tree or a rock or a table. In the world there might be seven rocks or seven trees or seven tables. But this is different than the pure number seven. You can find seven things, but this isn't the same as finding seven. So I'd say that, in some sense, the number seven is "imaginary," in much the same way that the number i is imaginary.

But you might be wondering if there are any physical manifestations of i. We could find seven rabbits. Could we ever find i rabbits? The answer to this question is no, at least as far as the rabbits are concerned. However, there are physical phenomena that are commonly described by complex numbers. One example is alternating current (AC) circuits. Determining the behavior of AC circuits with resistors, capacitors, and inductors is very commonly done with complex numbers. And the theory of quantum mechanics also makes extensive use of complex numbers.

In any event, I'll typically use z to refer to a complex number. (This is standard, but not universal. Many authors use x or some other letter.) A generic complex number is a combination of real numbers and imaginary numbers of the following form:

$$z = a + bi \ , \tag{19.4}$$

where a and b are real numbers. For example, we might have:

$$z = 3 + 4i \ . \tag{19.5}$$

The numbers a and b in Eq. (19.4) are referred to as, respectively, the **real part** and the **imaginary part** of z. The real part of z is sometimes denoted $\Re(z)$, and the imaginary part is denoted $\Im(z)$. I won't use these notations in this book, but it's possible that you encounter them elsewhere.

Note that the imaginary part of z is not imaginary. I.e., the imaginary part of $z = 3 + 4i$ is 4. We would not say that the imaginary part is $4i$.

19.2 The Algebra of Complex Numbers

We now consider performing basic mathematical operations with complex numbers: addition, subtraction, and multiplication. The two keys are to treat i as if it were a separate algebraic variable, and to make use of the fact that $i^2 = -1$.

19.2.1 Addition and Subtraction

To add two complex numbers, one adds their real and imaginary parts separately. For example, let

$$z_1 = a + bi , \tag{19.6}$$

and

$$z_2 = c + di . \tag{19.7}$$

Then

$$z_1 + z_2 = (a + bi) + (c + di) = (a + c) + (b + d)i . \tag{19.8}$$

So the real part of $z_1 + z_2$ is just the sum of the real parts of z_1 and z_2, and similarly for the imaginary parts.

19.2.2 Multiplication

To multiply two complex numbers one multiplies the two numbers together as you would for binomials,[1] and then simplifies using the fact that $i^2 = -1$. Suppose we want to multiply together z_1 and z_2, given above in Eqs. (19.6) and (19.7):

$$
\begin{aligned}
z_1 z_2 &= (a + bi)(c + di) & (19.9)\\
&= ac + adi + bci + bdi^2 & (19.10)\\
&= ac + (ad + bc)i + bdi^2 & (19.11)\\
&= ac + (ad + bc)i - bd & (19.12)\\
&= (ac - bd) + (ad + bc)i . & (19.13)
\end{aligned}
$$

Note that in Eq. (19.12) I've used the fact that $i^2 = -1$.

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[1]You might have learned this process as FOILing. "FOIL" is a mnemonic device for accounting for the four terms that arise when multiplying binomials: Firsts, Outers, Inner, Lasts.

Let's consider a numerical example. Suppose we want to multiply together the numbers $4 + 3i$ and $2 - 3i$:

$$
\begin{aligned}
(4 + 3i) \times (2 - 3i) &= 8 + (3i)(-3i) + 4(-3i) + (3i)(2) & (19.14) \\
&= 8 - 9i^2 - 12i + 6i & (19.15) \\
&= 8 - 9(-1) - 6i & (19.16) \\
&= 17 - 6i \ . & (19.17)
\end{aligned}
$$

19.3 The Geometry of Complex Numbers

The real number line is a useful way of representing and visualizing the set of real numbers. We've made us of this when drawing phase portraits in previous chapters. There is a similar geometric way of visualizing complex numbers: **the complex plane**. The complex plane is a natural generalization of the real number line. We plot the real part of a complex number on the horizontal axis and the imaginary part on the vertical axis.

This is illustrated in Fig. 19.1. There are four numbers plotted in this figure, each is shown with a different shape. The circle is $2 + 2i$, the triangle is $\frac{1}{2} - \frac{3}{2}i$, the square is $-i$, and the diamond is -2.5. I'd suggest taking a moment to examine Fig. 19.1 to make sure that you see the connection between the real and imaginary parts of a complex number and its representation on the complex plane.

There is another way of indicating the location of complex numbers on a plane. Rather than specifying a number's real and imaginary parts, a and b, we specify how far the point is from the origin, and the angle between the horizontal axis and a line drawn from the origin to the point z. This is illustrated in Fig. 19.2. In the figure the complex number $z = a + bi$ is plotted as a small circle. The distance from the origin to the point is denoted r. The angle θ is the angle between the horizontal axis and the line from the origin to the point. So, we can specify the complex number by giving the real and imaginary parts, a and b, or by giving its r and θ values. The a, b form is known as Cartesian coordinates; the r, θ form is known as polar coordinates. Both forms are equivalent; they're simply different representations of the same number. Which form we use will depend on the situation; in some cases Cartesian coordinates are easiest, while in other cases polar coordinates are more convenient.
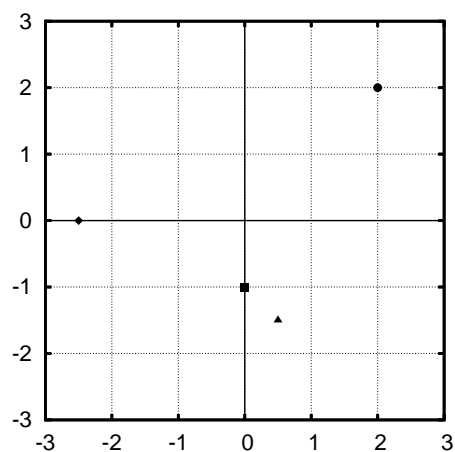
Figure 19.1: The complex plane. Four numbers are plotted: $2 + 2i$ (circle); $\frac{1}{2} - \frac{3}{2}i$ (triangle): -2.5 (diamond); and $-i$ (square).
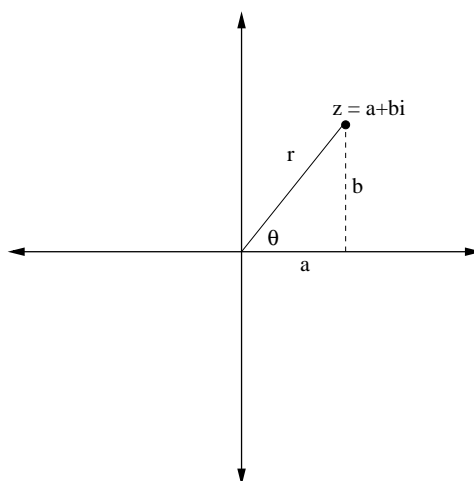
Figure 19.2: Illustrating polar notation for points $z = a + bi$ on the complex plane.

One can come up with equations that relate the two different forms for a complex number. For example, suppose we know a and b in Fig. 19.2 and we want to figure out r. Notice that the lengths a, b, r form a right triangle, with r the hypotenuse. Thus, by the Pythagorean theorem:

$$r^2 = a^2 + b^2 . \tag{19.18}$$

Solving for r, we get:

$$r = \sqrt{a^2 + b^2} . \tag{19.19}$$

We will use this equation often.

As an example, consider the point shown as a circle on Fig. 19.1. This point is $2 + 2i$, so a and b are both 2. Thus, the r for this point is

$$r = \sqrt{2^2 + 2^2} = \sqrt{8} \approx 2.83 . \tag{19.20}$$

Geometrically, this means that the pointed indicated by the circle is around 2.83 units away from the origin. The θ for this point is 45 degrees. To see this, draw a line from the origin to the point. The angle can then be seen to 45.

If one knows a and b one can also determine the angle θ. To do so requires trigonometry. The resulting formula is:

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) . \tag{19.21}$$

We won't use this equation much, if at all, in this version of the book. However, I include it here for the sake of completeness.

Finally, one can also use trigonometry to go from r and θ to a and b. The formulas are:

$$\begin{aligned} a &= r\cos\theta , & \text{(19.22)} \\ b &= r\sin\theta . & \text{(19.23)} \end{aligned}$$

We won't use these equations in this version of the book, but it's good to know that the exist.

19.4 The Geometry of Multiplication

Polar coordinates are particularly helpful when we are multiplying two complex numbers. We shall see that multiplication has a particularly simple

interpretation in polar coordinates. To do so, let's consider a series of examples. Our goal will be to use these examples to come up with a rule that will let us perform multiplication using polar coordinates instead of Cartesian.
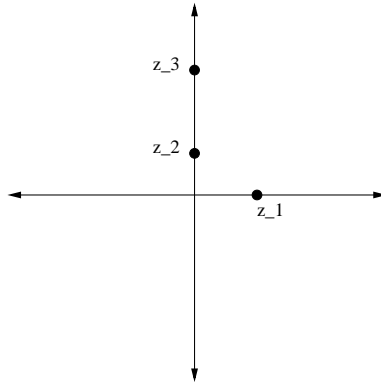

Figure 19.3: Illustrating multiplication; $z_1 = 3$, $z_2 = 2i$, and $z_3 = z_1 z_2 = 6i$.

First, let

$$z_1 = 3 , \tag{19.24}$$

and

$$z_2 = 2i . \tag{19.25}$$

These two numbers are shown on the complex plane in Fig. 19.3. Note that for z_1, $r = 3$ and $\theta = 0$, while for z_2, $r = 2$ and $\theta = 90$ degrees. Let's multiply these two numbers together and call this new number z_3:

$$z_3 = z_1 z_2 = 3 \times 2i = 6i . \tag{19.26}$$

The number z_3 is also shown in Fig. 19.3. In polar coordinates, z_3 is given by $r = 6$ and $\theta = 90$. Note that the r for z_3 is just the r for z_1 times the r for z_2. And the θ for z_3 is the same as the θ for z_2.

For our next example, consider the following two numbers

$$w_1 = 2 + 2i , \tag{19.27}$$

and

$$w_2 = -1 + i . \tag{19.28}$$

(I'm using w for complex numbers here to distinguish them from the z's of the previous example.) These numbers are shown in Fig. 19.4. Note that for
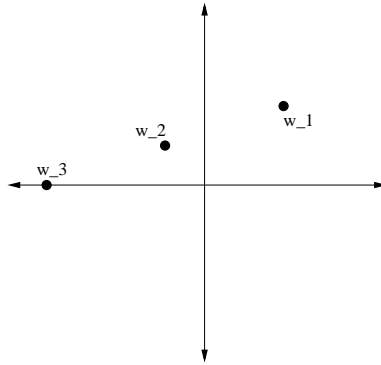
Figure 19.4: Illustrating multiplication; $w_1 = 2+2i$, $w_2 = -1+i$, and $w_3 = 4$.

w_1, $r_1 = \sqrt{8} \approx 2.83$ and $\theta_1 = 45$ degrees. To determine r, I used Eq. (19.19). And for w_2, $r_2 = \sqrt{2} \approx 1.41$ and $\theta_2 = 135$ degrees.

Let's multiply these two numbers together and see what happens:

$$w_3 = w_1 w_2 = (2+2i)(-1+i) = -2+2i-2i+2i^2 = -2-2 = -4 . \quad (19.29)$$

Note that in polar coordinates, w_3 has $r_3 = 4$ and $\theta_3 = 180$. As was the case in the previous example, the r for w_3 is equal to the r's for w_1 and w_2 multiplied together:

$$r_3 = r_1 r_2 = \sqrt{8}\sqrt{2} = \sqrt{16} = 4 . \quad (19.30)$$

And the θ's get added:

$$\theta_3 = \theta_1 + \theta_2 = 45 + 135 = 180 . \quad (19.31)$$

This is a general result. *To multiply two complex numbers, multiply their r's and add their θ's.* This result gives us a way to multiply complex numbers that is often faster than using Cartesian coordinates. More important, it gives us a geometric way of "seeing" what the effects of multiplication are. Practicing multiplying and plotting complex numbers in the problems below should help to generate some intuition about the effects of multiplication.

Finally, note that the above result for multiplying in polar coordinates leads to a simple view of squaring complex numbers. Squaring just means multiplying a number by itself:

$$z^2 = zz . \quad (19.32)$$

Thus, to square a complex number, all one has to do is square the r value and double the θ value. For example, suppose $z = 1 + i$, so $r = \sqrt{2}$ and $\theta = 45$. Then, z^2 has an r of $(\sqrt{2}) = 2$ and a θ of 90. A θ of 90 indicates that the point is located "straight up." That is, the point is on the vertical (imaginary) axis. So it must be that $z^2 = 2i$.
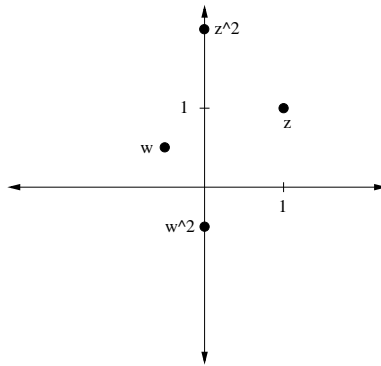

Figure 19.5: Illustrating squaring complex numbers; $z = 1 + i$, $z^2 = 2i$, $w = \frac{1}{2} + \frac{1}{2}i$, $w_3 = 4$.

We can verify this result by using Cartesian coordinates:

$$z^2 = (1+i)^2 = (1+i)(1+i) = 1 + i + i + i^2 = 1 + 2i - 1 = 2i , \quad (19.33)$$

which is the same as our result obtained via Cartesian coordinates.

Squaring z is illustrated in Fig. 19.5, where I've plotted z and z^2 on the complex plane. Note that we can see that the r for z^2 is equal to the square of the r of z, and that the θ for z^2 is twice that of z.

Also on Fig. 19.5 I've shown $w = -\frac{1}{2} + \frac{1}{2}i$ and w^2. The θ for w is 135, while r is given by:

$$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{2}{4}} = \frac{1}{\sqrt{2}} \approx 0.707 . \quad (19.34)$$

Thus, w^2 has an r given by the square of this amount. Namely, $r = \frac{1}{\sqrt{2}^2} = \frac{1}{2}$. Note that the r of w^2 is less than the r of w. The θ for w^2 is twice that of w: namely, $\theta = 270$. So, w^2 has an r of $\frac{1}{2}$ and a θ of 270. This is shown in Fig. 19.5.

19.5 Problems

For problems 1–4 let

$$z_1 = 3 , \tag{19.35}$$

$$z_2 = 2 - i , \tag{19.36}$$

$$z_3 = 4i , \tag{19.37}$$

and

$$z_4 = -2 + 3i . \tag{19.38}$$

1. Calculate the following quantities:

   (a) $z_1 + z_2$

   (b) $z_1 + z_3$

   (c) $z_1 + z_4$

2. Calculate the following quantities, and plot the two numbers and their product on the complex plane.

   (a) $z_1 z_2$

   (b) $z_4 z_2$

   (c) $z_2 z_4$

3. Calculate the following quantities:

   (a) $5 z_4$

   (b) $z_4 + 5$

   (c) $(z_3 - 1)(z_1 = 2i)$

   (d) $(z_3 + z_4) z_2$

4. Calculate the following quantities:

   (a) z_4^2

   (b) z_3^3

   (c) z_1^2

   (d) z_2^2

5. Consider the function $f(z) = z^2$. Calculate the first 3 iterates of

   (a) $z_0 = 0$
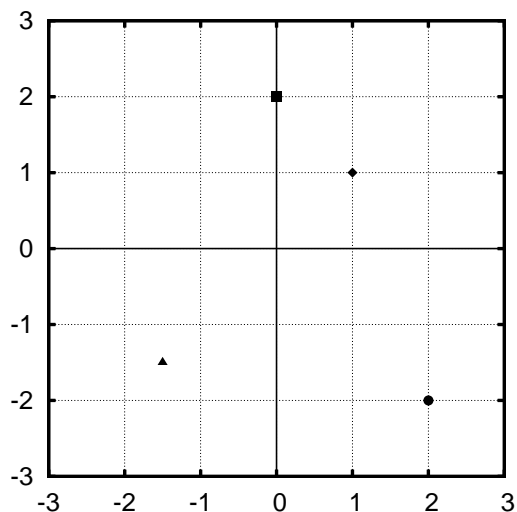   (b) $z_0 = i$
   (c) $z_0 = 2$

Figure 19.6: Four complex numbers plotted on the complex plane. See question 6.

6. Four complex numbers are shown on the complex plane in Fig. 19.6. Determine the coordinates for each point, using both Cartesian and polar coordinates.

7. (Important!) Consider the squaring function $f(z) = z^2$. For each of the initial conditions listed below, do the following.

   • Convert to polar representation.
   • Calculate the first several iterates.
   • State what you think the long-term behavior of the orbit is

   (a) $z + 0 = i$
   (b) $z + 0 = 1 - i$

(c) $z + 0 = 2 + i$

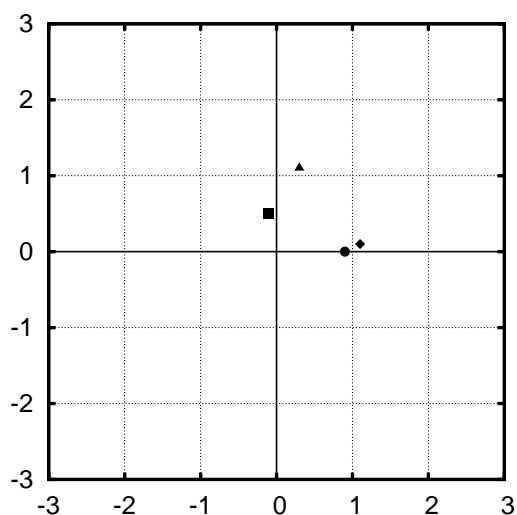(d) $z + 0 = 2$

(e) $z + 0 = -1$

(f) $z + 0 = -1 + 2i$

Figure 19.7: Four different initial conditions plotted on the complex plane. See question 8.

8. In Fig. 19.7 are shown four different complex numbers. Use each as a seed for the squaring function $f(z) = z^2$. On the complex plane, sketch the first three iterates for each seed. Perform the iteration using the geometric view of squaring complex numbers.

9. What do you think the Julia set is for the function $f(z) = z^2$, where z is a complex number? That is, what initial conditions z_0 have the property that, when iterated by f, they do not fly off to infinity?

10. Consider the function $f(z) = z^2 - 1$. Calculate the first 3 iterates of

(a) $z_0 = 0$

(b) $z_0 = i$

(c) $z_0 = 2$

11. Consider the function $f(z) = z^2 + i$. Calculate the first 3 iterates of

   (a) $z_0 = 0$

   (b) $z_0 = i$

   (c) $z_0 = 2$