Complex Numbers

The number i is defined as the square root of negative one:

$$i \equiv \sqrt{-1}\ . \tag{1}$$

Thus,

$$i^2 = -1\ . \tag{2}$$

General complex numbers are a combination of real and imaginary numbers. For example, a generic complex number z can be written as:

$$z = a + bi\ . \tag{3}$$

We refer to a as the real part of z and b as the imaginary part of z. Note that the imaginary part is not an imaginary number.

Addition and Subtraction: To add and subtract complex numbers, we add (or subtract) the real and imaginary parts separately. For example:

$$(4 + 3i) - (2 - 3i) = (4 - 2) + (3 - -3)i = 2 + 6i\ . \tag{4}$$

Multiplication: To multiply two complex numbers, "FOIL" as you would for binomials and then simplify using $i^2 = -1$. For example:

$$\begin{aligned}(4 + 3i) * (2 - 3i) &= 8 + (3i)(-3i) + 4(-3i) + (3i)(2) \\ &= 8 - 9i^2 - 12i + 6i \\ &= 8 - 9(-1) - 6i \\ &= 17 - 6i \end{aligned} \tag{5}$$

Polar Form

It's conventional and convenient to picture complex numbers as points on a plane, with the x-coordinate given by the real part of the number, and the y-coordinate by the imaginary part. It is often easier (really!) to give the point on the plane in polar coordinates: i.e. by specifying the point's distance r from the origin and its angle θ from the horizontal. The two different forms for a complex number are related by the following equations:

$$a = r\cos\theta \, , \tag{6}$$

$$b = r\sin\theta \, , \tag{7}$$

and,

$$r = \sqrt{a^2 + b^2} \, , \tag{8}$$

$$\theta = \tan^{-1}(\frac{b}{a}) \, , \tag{9}$$

I strongly suggest always drawing a set of axes and labeling angles and such to avoid confusion with minus signs.

In quantum mechanics, we will very often write complex numbers using exponential forms. The key step here is the following relationship:

$$e^{ix} = \cos x + i\sin x \, . \tag{10}$$

This means that if we have the r and the θ for a complex number z, then we may write it as:

$$z = re^{i\theta} = r\cos\theta + i\sin\theta \, . \tag{11}$$

We will frequently need to take the complex conjugate z^* of the complex number z. This is done by placing a minus sign in front of all i's in z. For example, if $z = a + ib$, then $z^* = z - ib$. Note that zz^* is real.