Homework assignment one

Due Friday September 13, 2002, 4:00 PM

1. Shannon Entropy.

   (a) Consider two random variables, X and Y. Suppose that they are independent. I.e., $\Pr(X,Y) = \Pr(X)\Pr(y)$. Show that $H[X,Y] = H[X]H[Y]$.

   (b) Consider the following distribution of a random variable X that can take one of three values, a, b, or c: $\Pr(X = a) = p_a = 1/4$, $\Pr(X = b) = p_b = 1/4$, and $\Pr(X = c) = p_c = 1/2$. Show that the grouping property of H holds:

   $$H[\{p_a, p_b, p_c\}] \; = \; H[\{p_a + p_b, p_c\}] \; + \; (p_a + p_b) \, H\left[\{\frac{p_a}{p_a + p_b}, \frac{p_b}{p_a + p_b}\}\right] \; . \tag{1}$$

   (c) A fair coin is flipped until the first head occurs. Let X denote the number of flips required. Find the entropy $H[X]$ in bits. Do do so, you'll need to recall a few standard facts about geometric series. (Cover and Thomas, Chapter 2, problem 1(a))

2. In this problem you will calculate the entropy per spin of a paramagnet several different ways. Consider a square $N \times N$ lattice. The spins don't interact with each other, but they do interact with an external field, B. The Hamiltonian of the system is:

   $$\mathcal{H} = \sum_{i=1}^{N} \sum_{j=1}^{N} -B S_{ij} \; , \tag{2}$$

   where $S_{ij} \in \{-1, +1\}$.

   (a) First, calculate the entropy by using the canonical ensemble. Find the free energy F and the energy E, and then determine the entropy S via $F = E - TS$.

   (b) Now, use the canonical ensemble to write down the probability that a single spin is up. Then determine the Shannon-Gibbs entropy of a single spin, being sure that you use a normalized distribution. Then use this to infer the entropy of the entire system.

(c) Determine S as a function of T by direct counting and using the microcanonical ensemble as we did in class.

(d) Show that, in the $N \to \infty$ limit, your answers for questions 2a, 2b, and 2c agree.

3. Show that the canonical distribution

$$p_i \propto e^{-\beta E_i} \tag{3}$$

minimizes the free energy $F = E - TS$. To show this, use the Shannon form of the entropy, and use

$$E = \sum_i p_i E_i \,. \tag{4}$$

4. Fun with counting and binomial coefficients.

(a) Five spin variables, $s_1, s_2, \ldots, s_5$, in a magnetic field are statistically independent. Each can take the values $\pm\hbar/2$ with the probabilities:

$$\Pr(s = \hbar/2) = p\,, \text{ and } \Pr(s = -\hbar/2) = 1 - p\,. \tag{5}$$

What is the probability that exactly three spins are up ($+\hbar/2$)? (Garrod, problem 1.2)

(b) Twelve books, containing a 4-volume series, are placed in random order on a shelf. What is the probability that the series is placed together and in order from left to right? (Garrod, problem 1.5)

5. A harmonic oscillator oscillates with amplitude A. If the time t is chosen at random, what is the probability that $a \le x(t) \le a + da$? (Garrod, problem 1.22)

6. Each second a particle, which was initially at $x = 0$, jumps either left or right a distance a, each with a probability of $\frac{1}{2}$. At time $t_n = n$ the particle is at location $x_k = ka$ with probability $\Pr(n, k)$. Calculate $\Pr(n, k)$ and show that, as n and k approach infinity, your result agrees with the central limit theorem. (I.e., show that $\Pr(n, k)$ is a Gaussian distribution.) (Garrod, problem 1.27) This problem is a little tricky. It's a standard illustration of the central limit theorem. As such, you should be able to find textbook references that will help.

7. The probability $W(n)$ that an event characterized by a probability p occurs n times in N trials was shown to be given by the binomial distribution:

$$W(n) = \frac{N!}{n!(N-n)!}\, p^n (1-p)^{N-n}\,. \tag{6}$$

Consider a situation where the probability p is small ($p \ll 1$) and where one is interested in the case $n \ll N$. (Note that if N is large, $W(n)$ becomes very small if $n \to N$ because of the smallness of the factor p^n when $p \ll 1$. Hence $W(n)$ is indeed only appreciable when $n \ll N$.) several approximations can then be made to reduce Eq. (6) to simpler form.

(a) Taylor expanding $\ln(1-p)$ for small p, show that $(1-p)^{N-n} \approx e^{-Np}$.

(b) Show that $N!/(N-n)! \approx N^n$.

(c) Finally, show that

$$W(n) = \frac{\lambda^n}{n!} e^{-\lambda} , \tag{7}$$

where $\lambda \equiv Np$ is the mean number of events. (Reif, problem 1.9)

8. Problem 5.24 from Chandler. This problem involves a figure which I'm not going to try and reproduce. Find a copy of Chandler and photocopy pp.155-6.